Attack of the Exponentials

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  1. $\neg X \vee \neg Y \vee X$;
  2. $(X \wedge Y \wedge \neg Z) \vee (X \wedge \neg Y) \vee \neg X \vee Z$.

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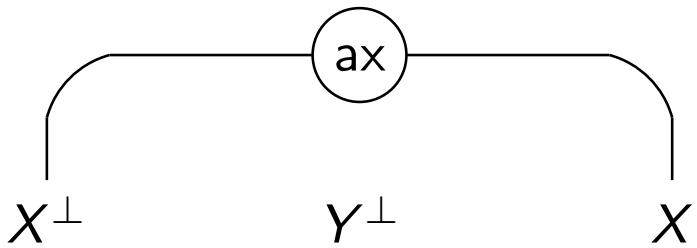

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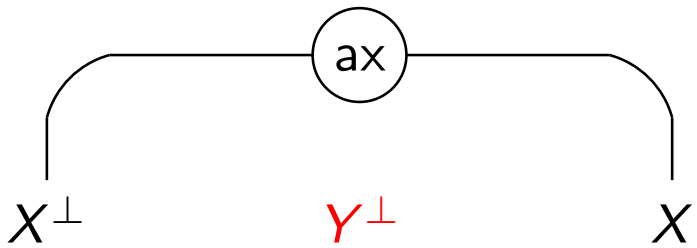

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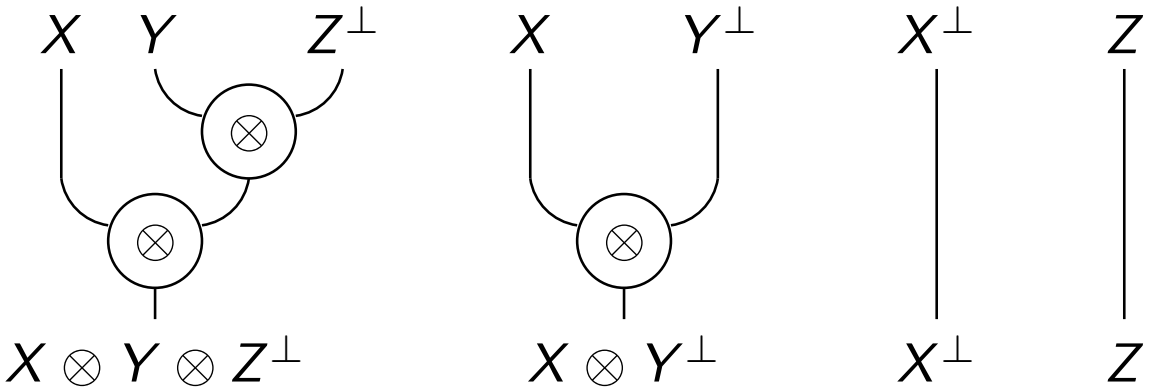

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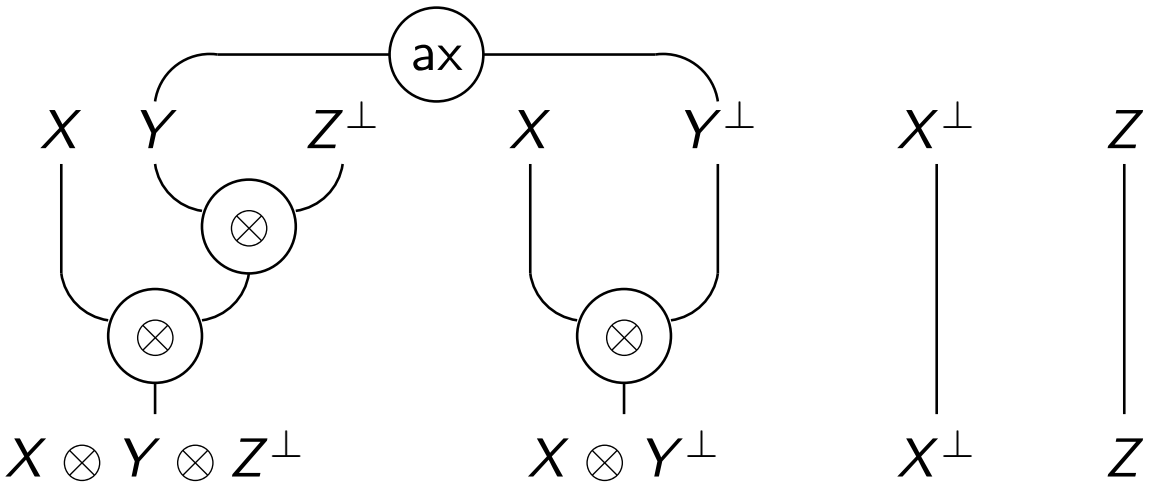

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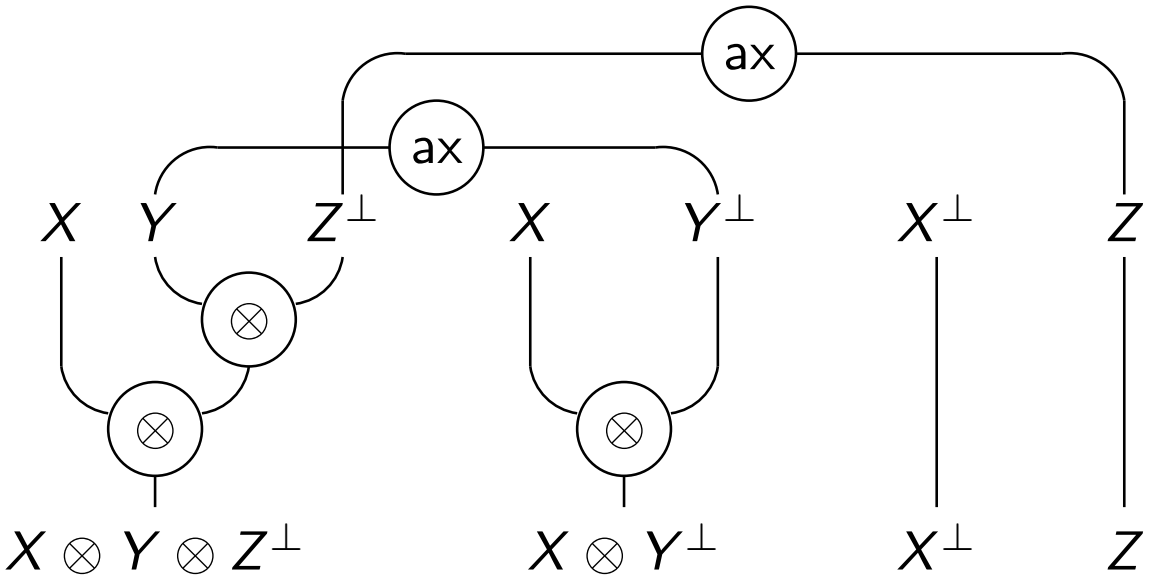

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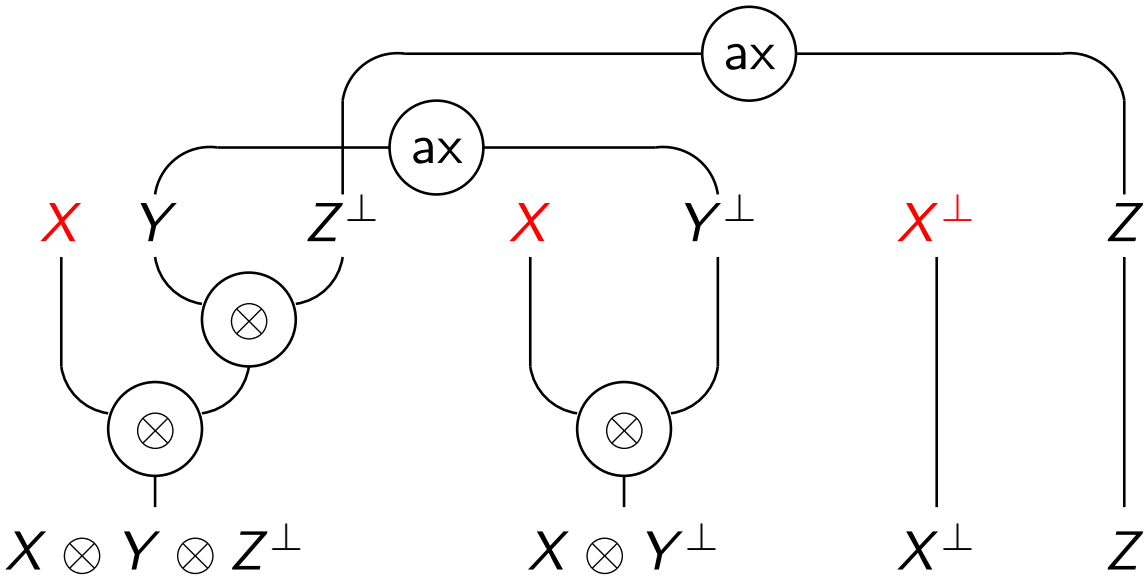

Structural rules

- Mathematical truth is *cumulative* and *inexhaustible*:

$$\frac{\Gamma \vdash C}{\Gamma, A \vdash C}\ \text{weakening} \qquad \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}\ \text{contraction}$$

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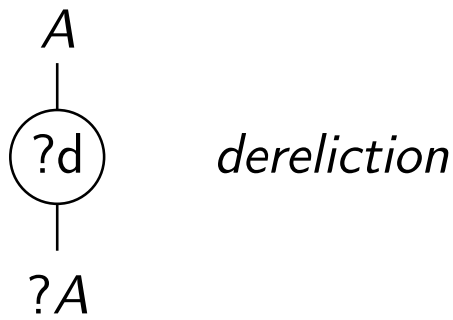
- Linear logic replaces truth with the notion of *resource*:
  - no structural rules on arbitrary formulas;
  - in other words, no arbitrary erasing and duplication:

$$A \not\multimap 1$$
$$A \not\multimap A \otimes A$$

The "why not" modality

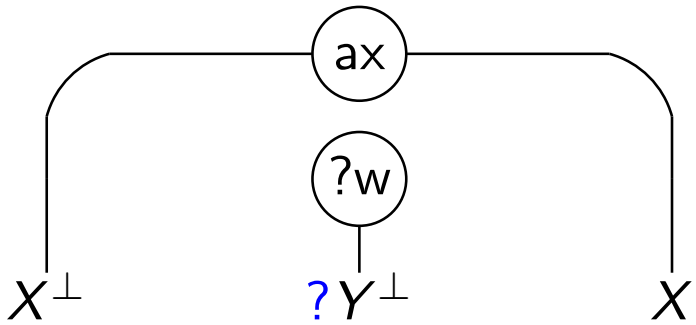
- We introduce a unary connective (*modality*) "why not", denoted by ?.
- Structural rules are allowed only on formulas of the form $?A$:

- We need to explicitly declare a formula as "contractible":



Revisiting classical tautologies

- Instead of $\neg X \vee \neg Y \vee X$, we prove $X^{\perp} \parr ?Y^{\perp} \parr X$:



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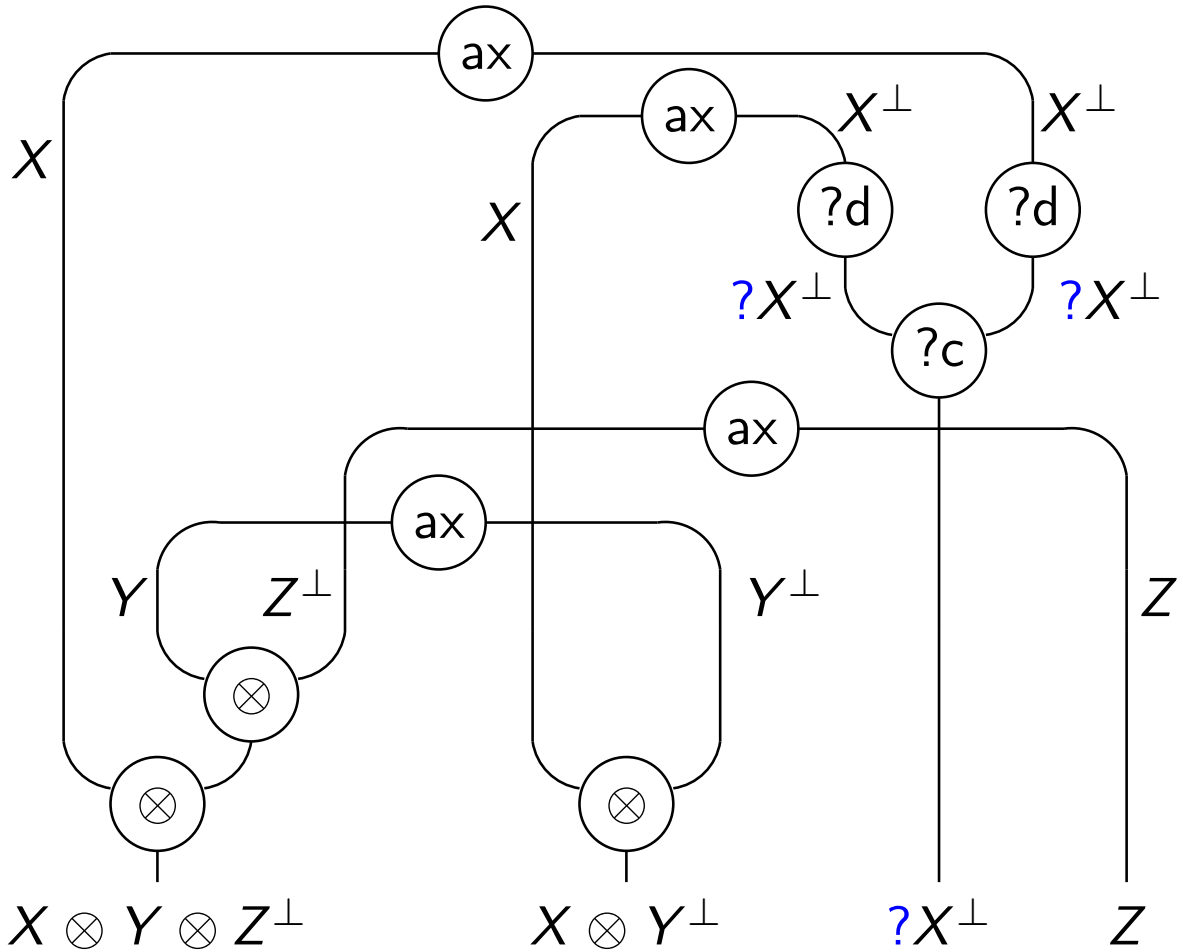


The meaning of linear sequents

- In classical logic, $\vdash \Gamma$ means "one of the formulas in Γ is necessarily true".
- This is because $\Gamma \vdash \Delta$ is $\bigwedge \Gamma \Rightarrow \bigvee \Delta$.
- In linear logic, $\Gamma \vdash \Delta$ is $\bigotimes \Gamma \multimap \parr \Delta$, with $A \multimap B := A^\perp \parr B$.
- What does that mean???

$A \otimes B$	simultaneous availability of both A and B
$A \multimap B$	A is needed to yield B (loosing A in the process)
$A \parr B$	$A^\perp$ is needed to yield B *and* $B^\perp$ is needed to yield A

- So, in linear logic, $\vdash A_1, \ldots, A_n$ means

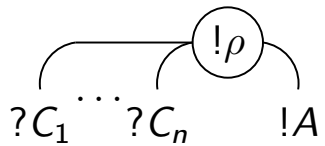
  for any $1 \leq i \leq n$, $A_1^\perp, \ldots, A_{i-1}^\perp, A_{i+1}^\perp, \ldots, A_n^\perp$ are needed to obtain A_i.

- The meaning of "why not" is best understood via duality:

$?A$	$A^\perp$ is needed an unspecified number of times
$!A$	A is available at will

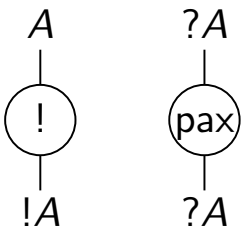
The "of course" modality

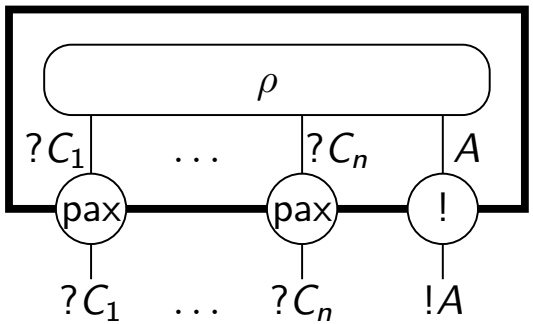
- The dual of "why not" is "of course", or "bang", denoted !.
- Two alternative presentations in nets:
  - **inductive:** nets are defined inductively on their *exponential depth*:
    - *depth* 0: as for MLL, with nodes ax, cut, $\otimes$, $\parr$, ?d, ?w and ?c;
    - *depth* $n > 0$: as above but also with nodes of the form



where ρ is a net of depth $< n$, of conclusions $?C_1, \ldots, ?C_n, A$.

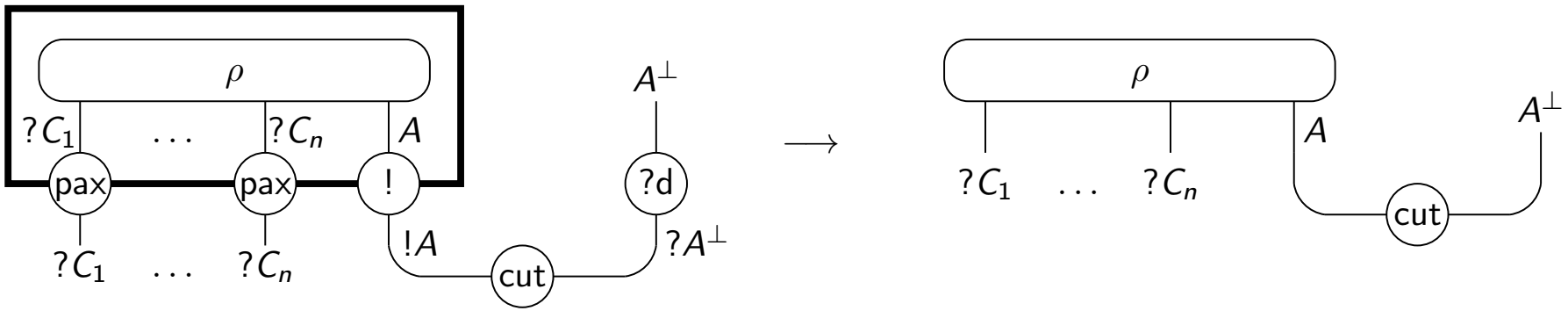
  - **global:** as depth 0 above but also with nodes ! and pax; each pax has an associated !, and each ! has an associated subnet, called *box*:





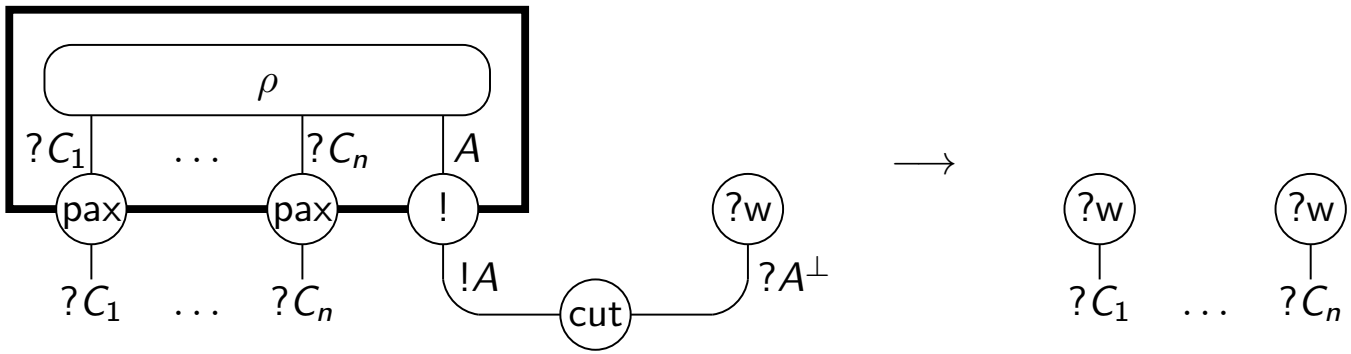
Boxes are either disjoint or included one in the other.

Cut-elimination: dereliction



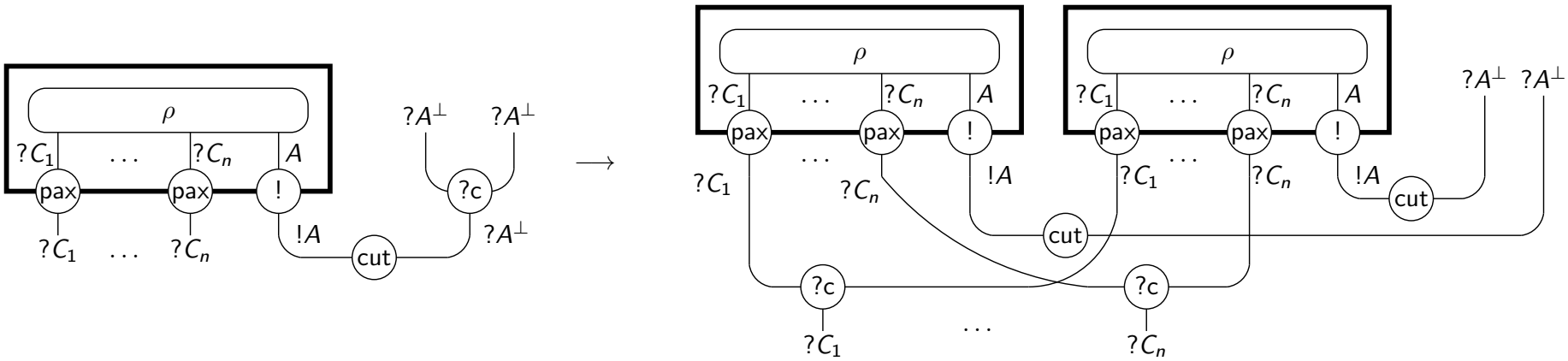
Dereliction "opens" a box.

Cut-elimination: weakening



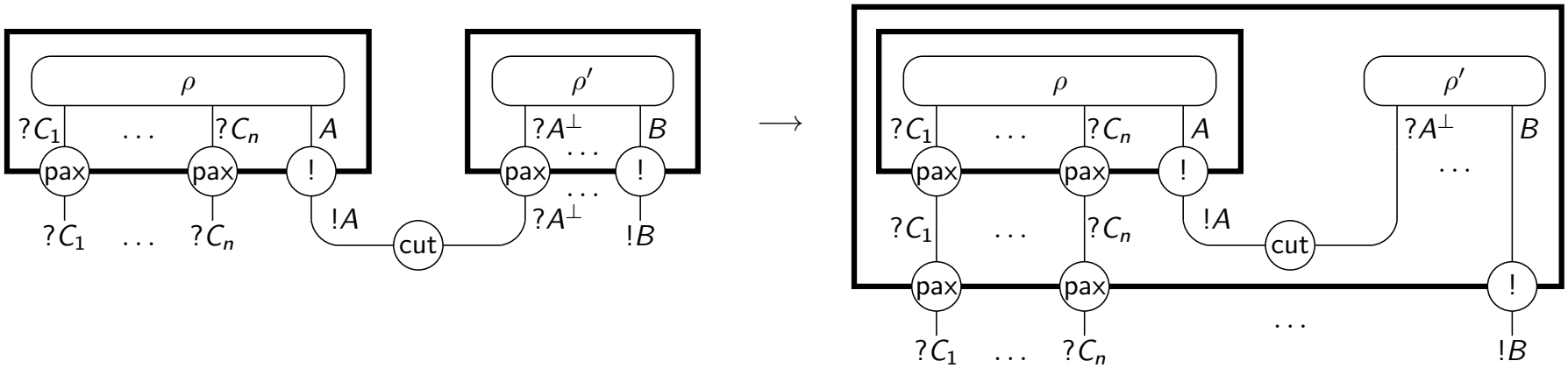
Weakening erases a box.

Cut-elimination: contraction



Contraction duplicates a box.

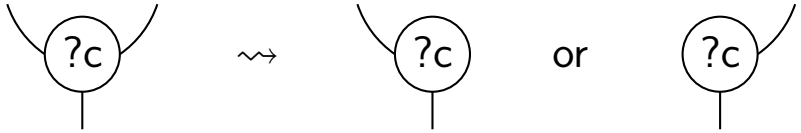
Cut-elimination: commutative step



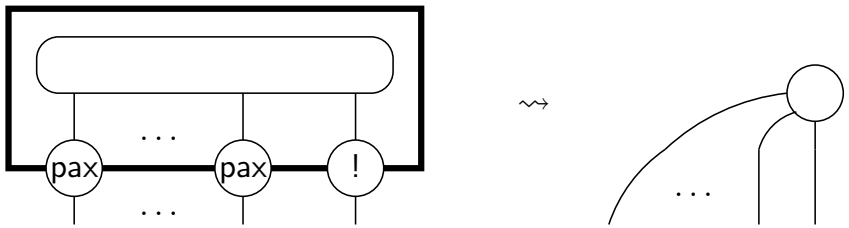
A box may "enter" inside another box.

Correctness

- Contractions are treated like $\parr$ nodes:



- Boxes are "collapsed" to a single node (that's already the case in the inductive formulation):



- A proof net is a net such that:
  - every switching (graph obtained as above) is acyclic;
  - the contents of every box is a proof net.
- Preserved by cut-elimination: ρ correct, $\rho \longrightarrow \rho'$ implies ρ' correct.
- Lack of connectedness causes a little technical problem...

Exponentials in sequent calculus

- Sequent calculus rules for the exponential modalities:

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ ?d \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ ?w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ ?c \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ !$$

- Exponential axioms ("storage laws"):

functoriality:	$!(A \multimap B) \multimap !A \multimap !B$	
dereliction:	$!A \multimap A$	(retrieve)
digging:	$!A \multimap !!A$	(indirection)
weakening:	$!A \multimap 1$	(discard)
contraction:	$!A \multimap !A \otimes !A$	(copy)

Categorically: $!(-)$ is a monoidal comonad and free coalgebras are comonoids.

- The exponential isomorphism: $!(A \,\&\, B) \cong !A \otimes !B$ (just like $2^{a+b} = 2^a \cdot 2^b$)
- Lack of connectedness in proof nets corresponds to mix:

$$\frac{\vdash \Gamma_1 \quad \dots \quad \vdash \Gamma_n}{\vdash \Gamma_1, \dots, \Gamma_n}\ \mathrm{mix}$$

Recovering intuitionistic and classical logic

- Intuitionistic logic is actually a *fragment* of linear logic (at any order):

$$\begin{array}{rcccccccc} A, B & ::= & X & !A \multimap B & \forall \xi.A & \exists \xi.!A & 0 & A \,\&\, B & !A \oplus !B \\ & & X & A \Rightarrow B & \forall \xi.A & \exists \xi.A & \bot & A \wedge B & A \vee B \end{array}$$

Theorem (Embedding of intuitionistic logic)

$\Gamma \vdash A$ is provable in LJ iff $!\Gamma \vdash A$ is provable in the above fragment of LL.

Categorically: intuitionistic logic is the Kleisli category of the comonad $!(-)$.

- Other translations of intuitionistic logic in linear logic exist (CbN vs. CbV).
- Classical logic may also be translated:

$$\begin{array}{rcl} X^+ := ?!X & \qquad & (\neg X)^+ := ?!?X^\perp \\ (A \vee B)^+ := A^+ \parr B^+ & & (A \wedge B)^+ := ?(!A^+ \otimes !B^+) \\ (\forall x.A)^+ := ?!\forall x.A^+ & & (\exists x.A)^+ := ?!?\exists x.!A^+ \end{array}$$

The principle is the generalized Gödel translation $A^F = (A \Rightarrow F) \Rightarrow F$, with $F = \bot$.

Example: the drinker's formula

- In any bar, there's someone such that, if he drinks, then everyone drinks:

$$F := \exists x(D(x) \Rightarrow \forall y\, D(y)).$$

- Proof: either everyone's drinking, or there's someone who is not (*MrSober*).
  - 1st case: $D(z) \Rightarrow \forall y\, D(y)$ is true for any z, anybody is the existential witness;
  - 2nd case: $D(\mathit{MrSober}) \Rightarrow \forall y\, D(y)$ is true, so *MrSober* is our witness.

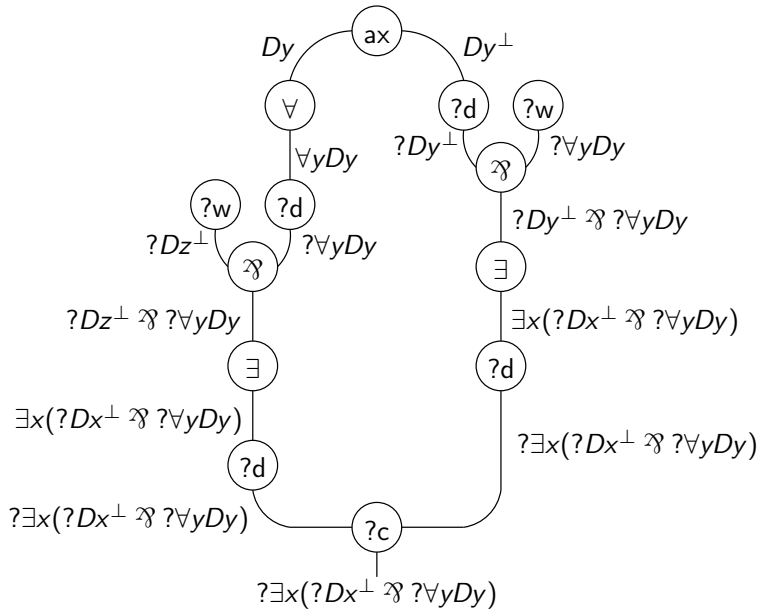
We used excluded middle (indeed, F is not provable intuitionistically).

In LK:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash Dy, \neg Dy}\,\mathrm{ax}}{\vdash Dy, \neg Dy, \forall y Dy}\,\mathrm{weak}}{\vdash Dy, \neg Dy \vee \forall y Dy}\,\vee}{\vdash Dy, F}\,\exists}{\vdash \forall y Dy, F}\,\forall}{\vdash \neg Dz, \forall y Dy, F}\,\mathrm{weak}}{\vdash \neg Dz \vee \forall y Dy, F}\,\vee}{\vdash F, F}\,\exists}{\vdash F}\,\mathrm{contr}$$

In LL:

$$F^+ \sim ?\exists x(?D(x)^\perp \parr ?\forall y D(y))$$



Curry-Howard: computation in linear logic

The Curry-Howard correspondence

logic	computer science
formula	type
proof	program
cut-elimination	execution

- Moral of the story: proof nets are programs!
- In particular: derivations of propositional NJ are simply-typed λ-terms.
- NJ (via LJ) is a fragment of LL $\Longrightarrow$ the λ-calculus embeds in proof nets.
- The decompostion $A \Rightarrow B = !A \multimap B$ appears at the level of execution:

$$(\lambda x.M)N \to M\{N/x\} \quad \text{vs.} \quad (\lambda x.M)N \to M[N/x] \to^* M\{N/x\}[N/x] \to M\{N/x\}.$$

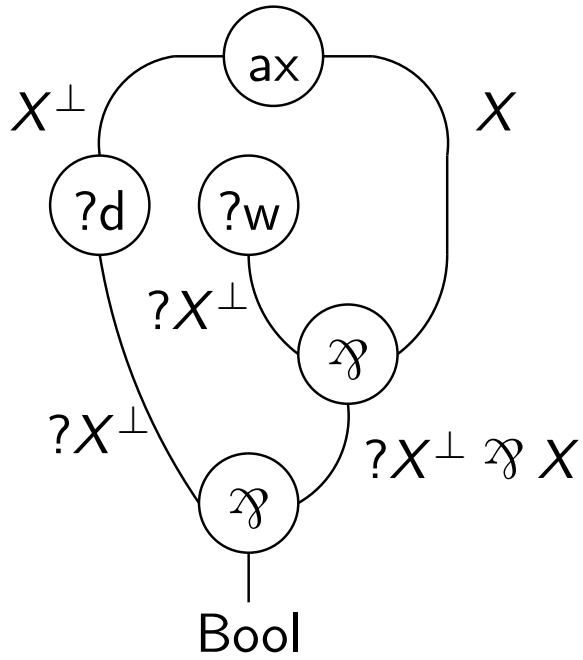
Example: Booleans and if... then... else

- Let

$$\text{Bool} \quad := \quad X \Rightarrow X \Rightarrow X \quad = \quad !X \multimap !X \multimap X \quad = \quad ?X^{\perp} \parr (?X^{\perp} \parr X).$$

- Let

$$\text{true} := \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X^{\perp}, X}\,\text{ax}}{\vdash ?X^{\perp}, X}\,?d}{\vdash ?X^{\perp}, ?X^{\perp}, X}\,?w}{\vdash ?X^{\perp}, ?X^{\perp} \parr X}\,\parr}{\vdash \text{Bool}}\,\parr \quad \rightsquigarrow$$



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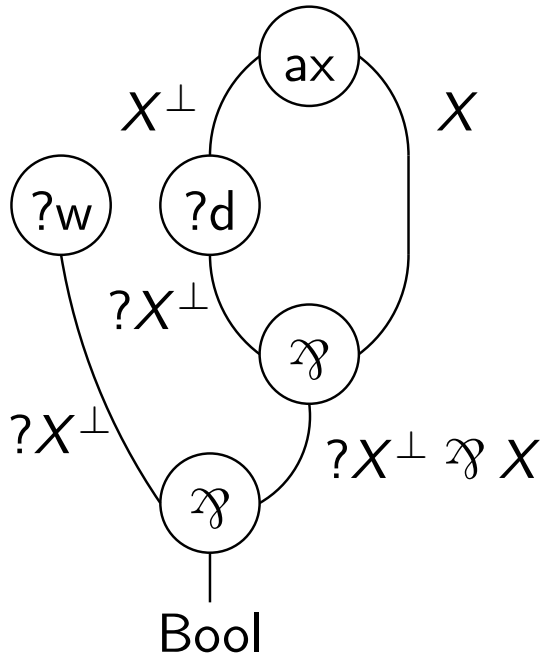

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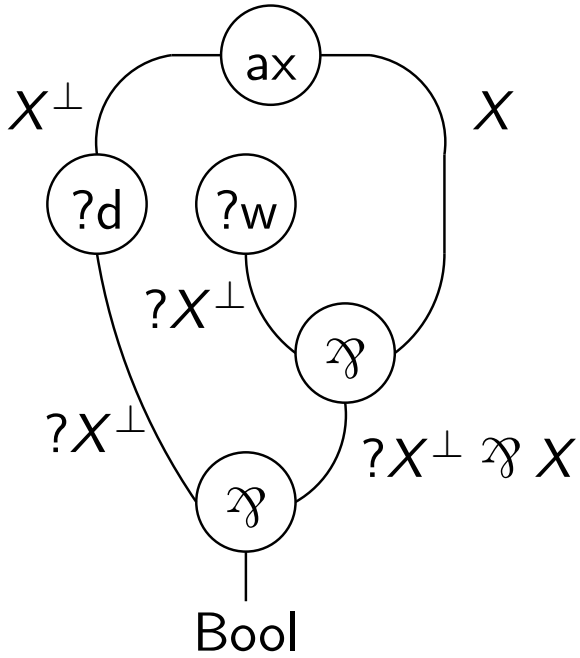

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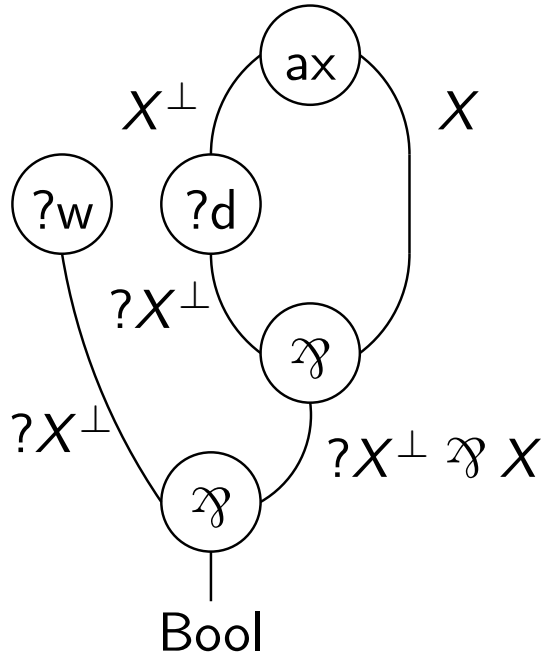


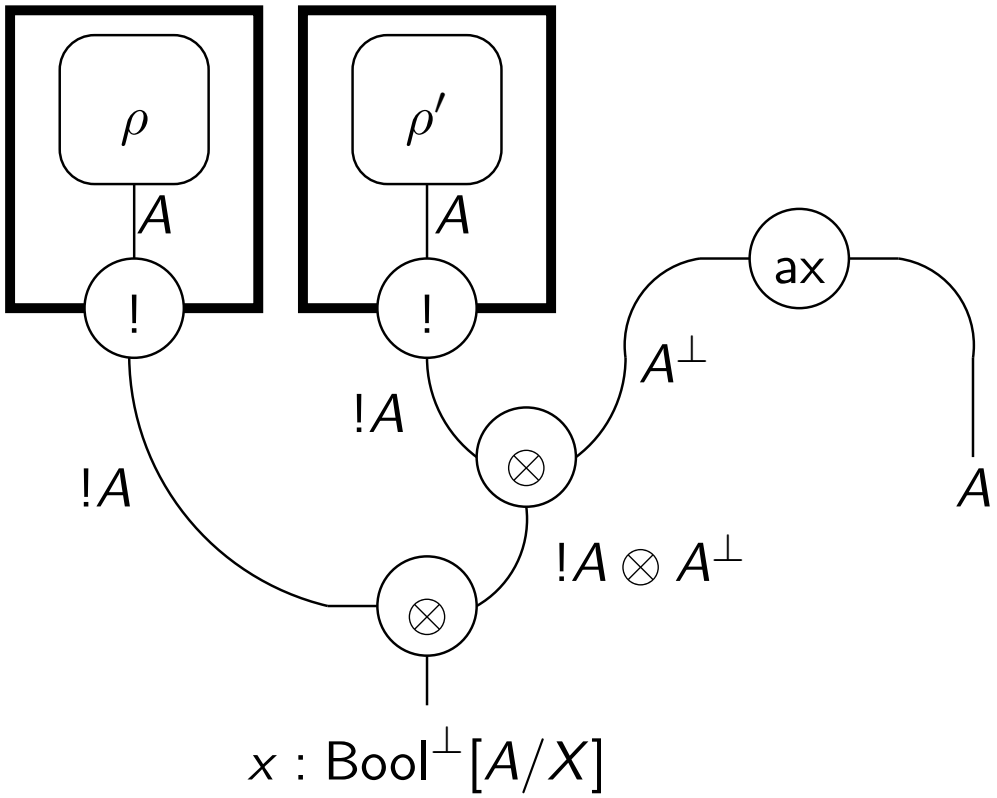
Example: Booleans and if. . . then. . . else

- We have

$$\mathsf{Bool}^{\perp} = !X \otimes (!X \otimes X^{\perp})$$

- Let $\rho, \rho' : A$ and let

if x then ρ else $\rho' :=$



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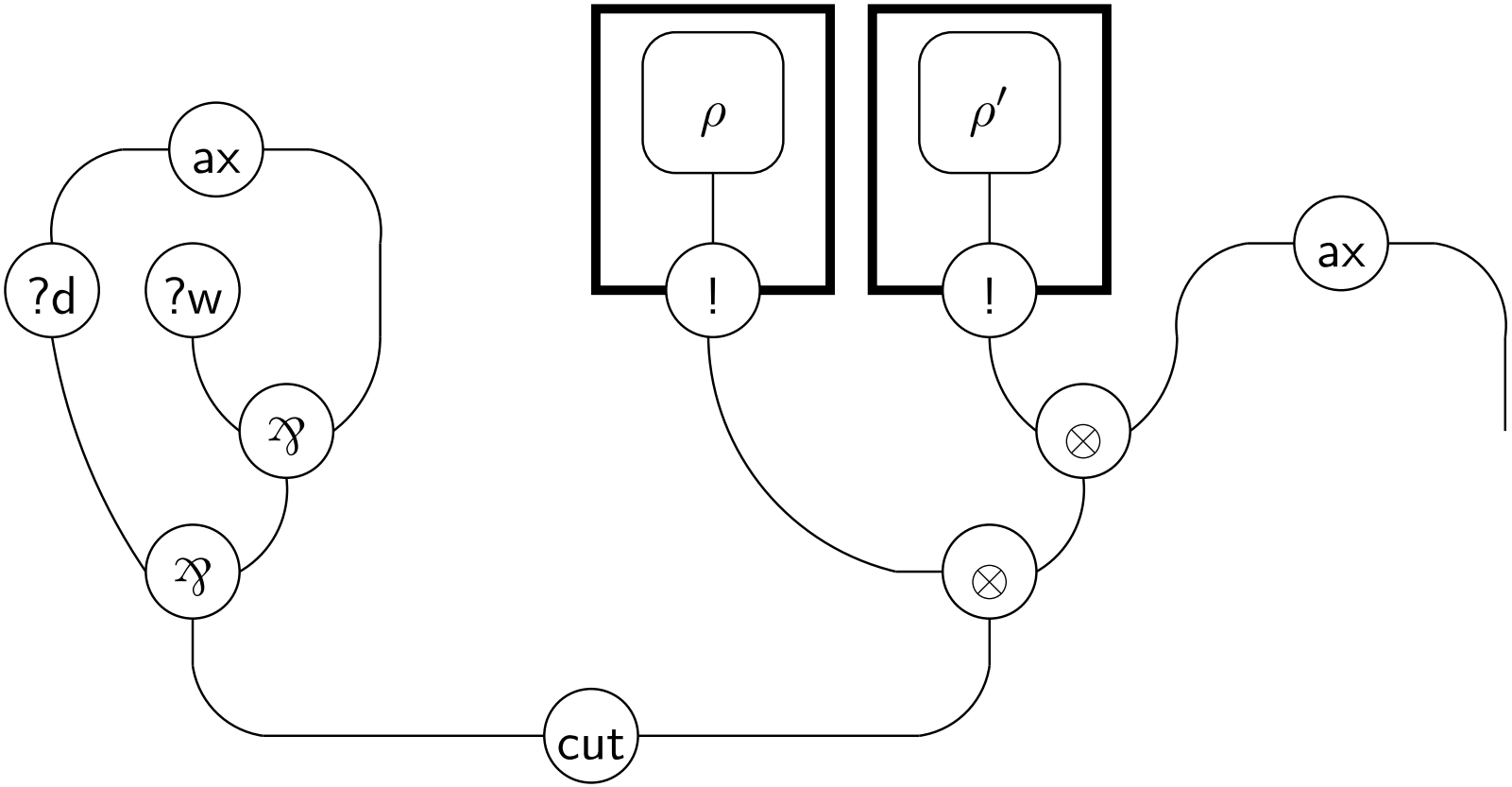
- Let us compute if $(\mathsf{true}[A/X])$ then ρ else ρ':



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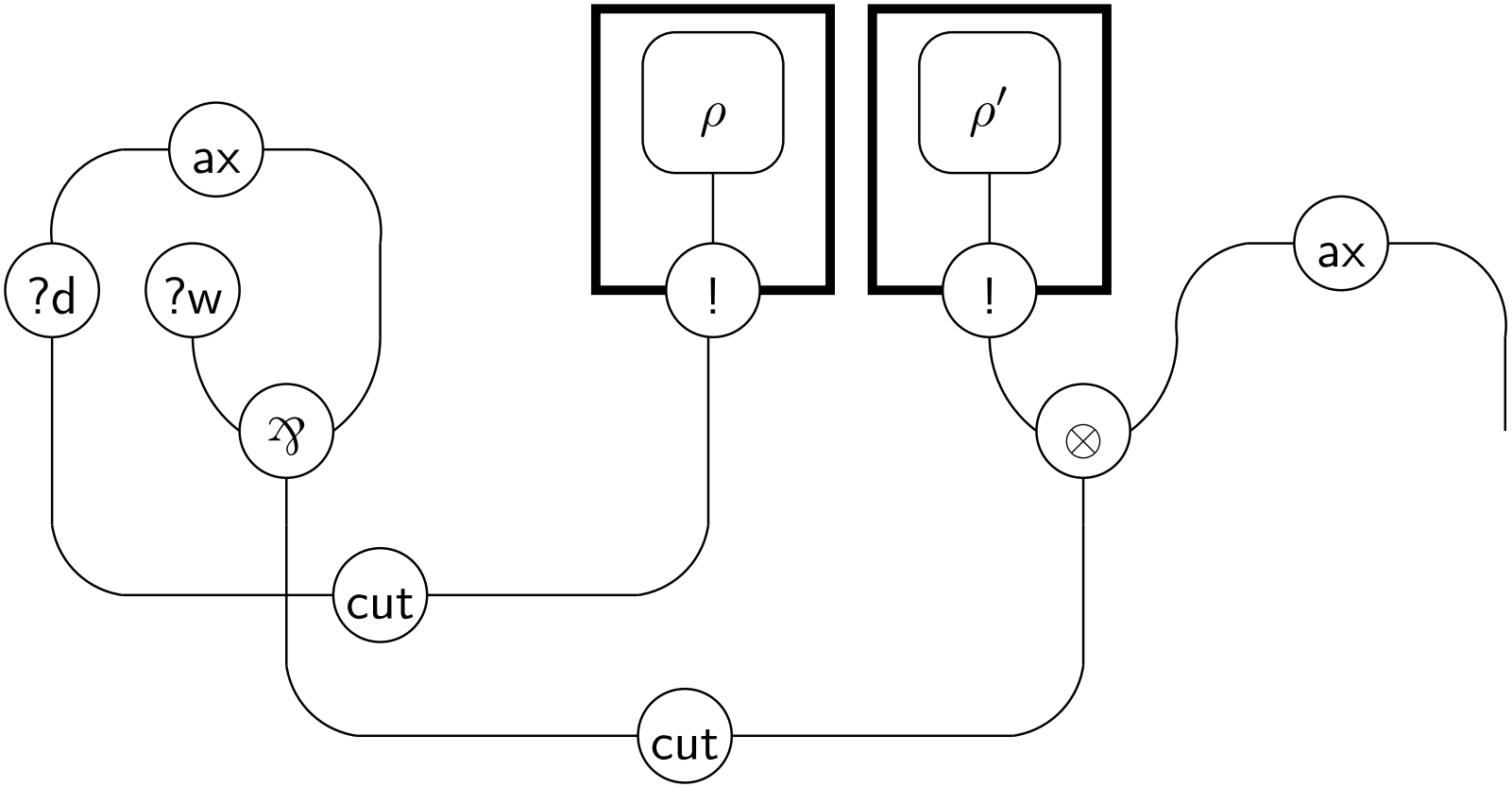

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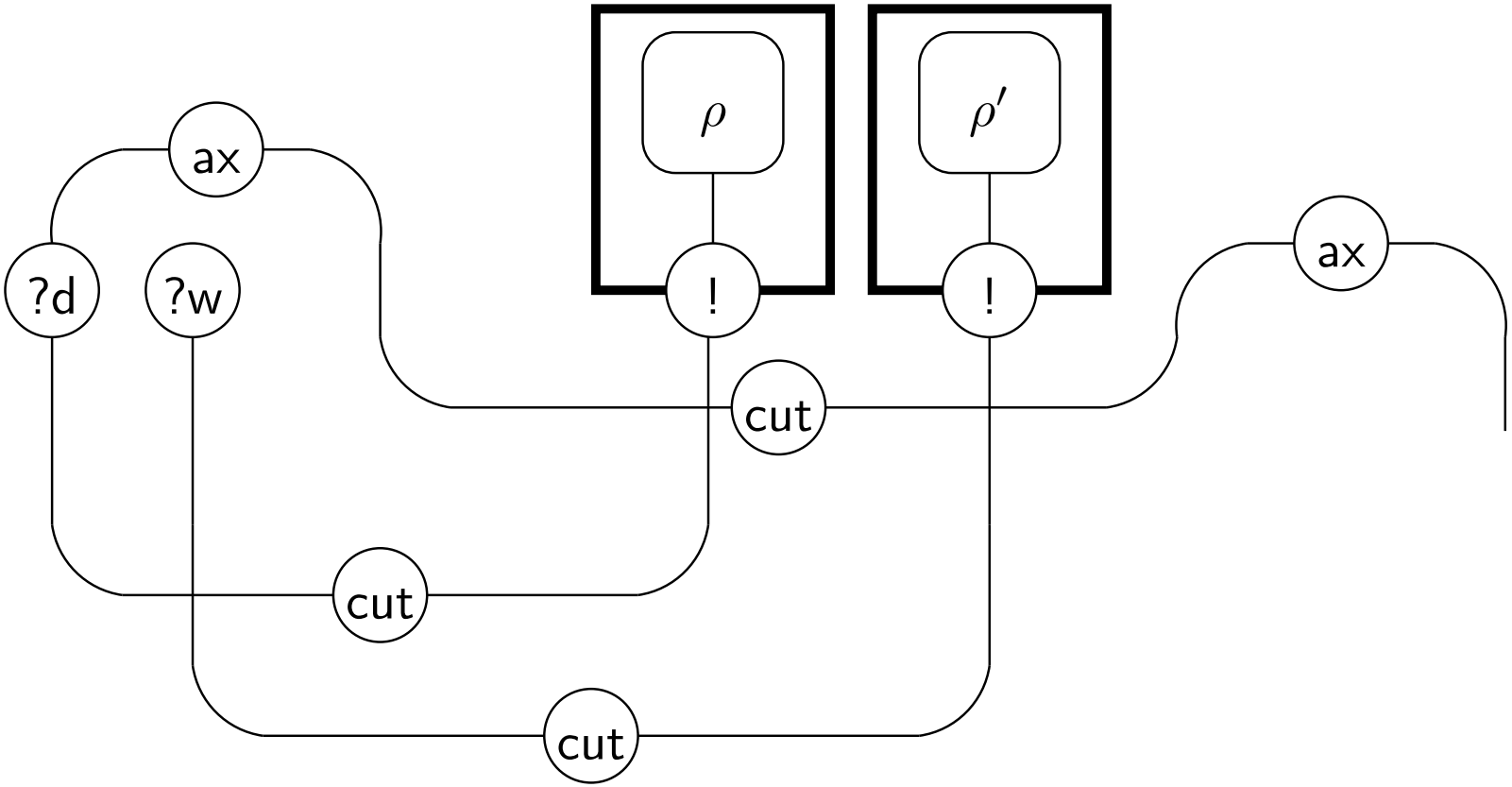

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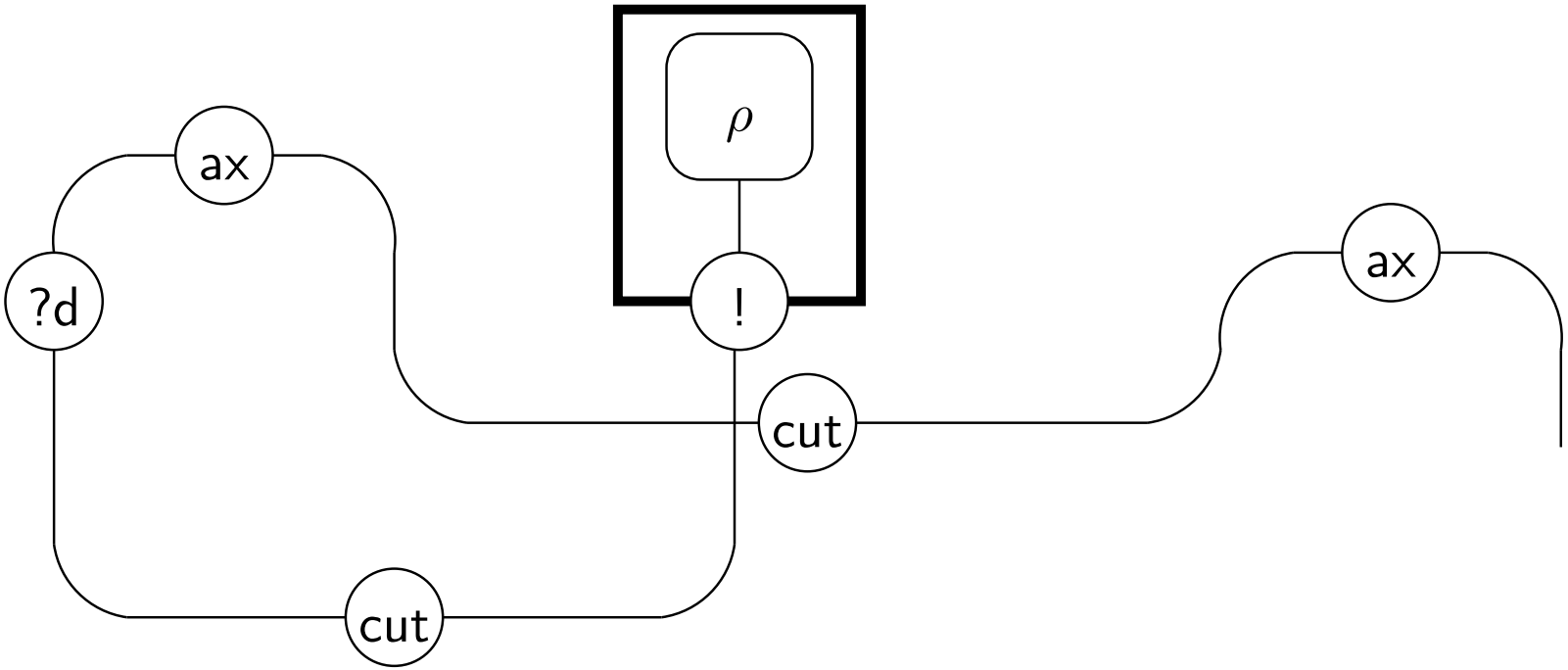

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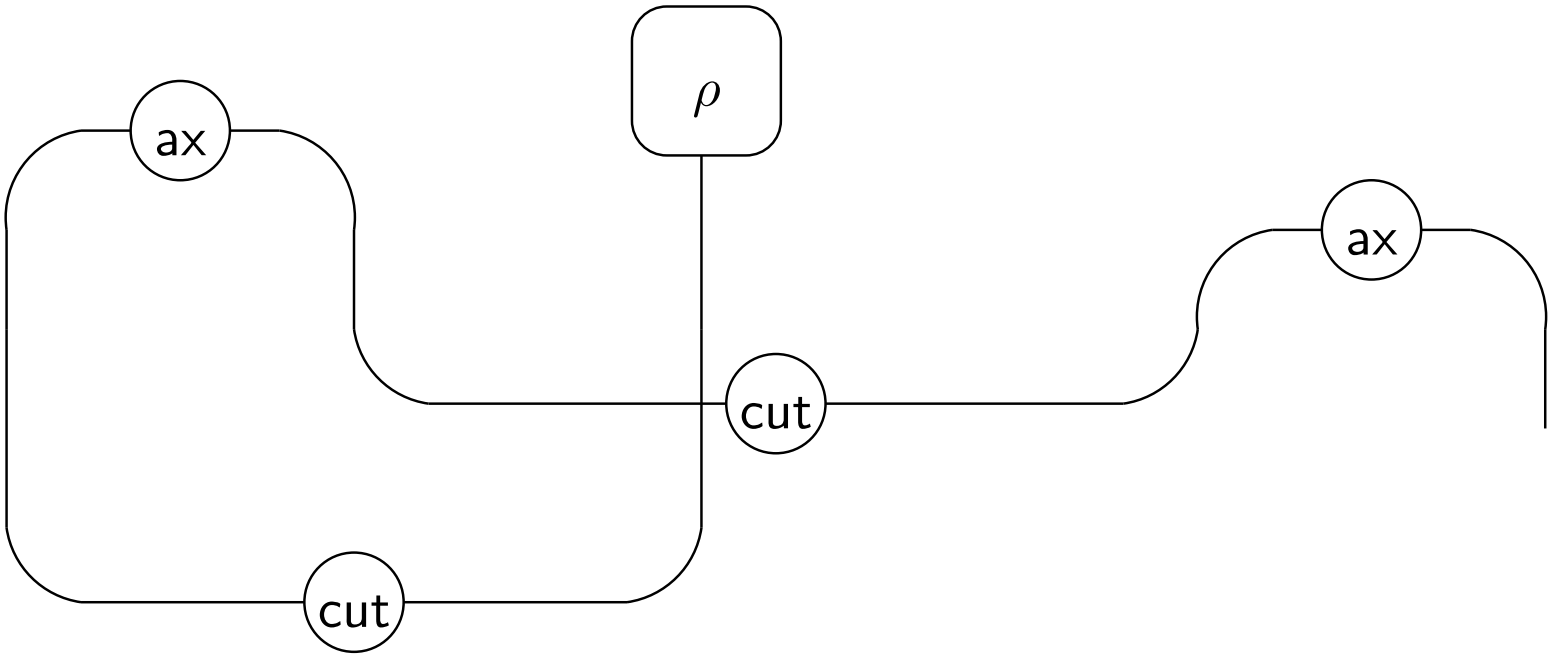

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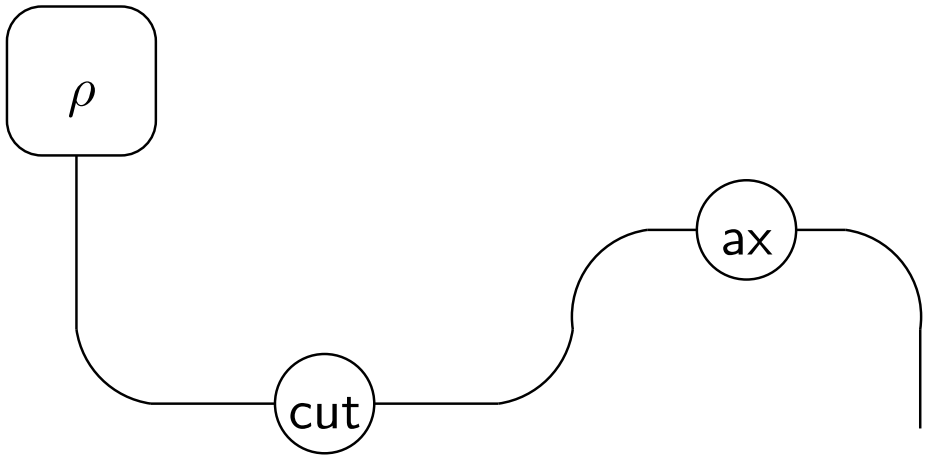
- Let us compute $\mathsf{if}\,(\mathsf{true}[A/X])\,\mathsf{then}\,\rho\,\mathsf{else}\,\rho'$:



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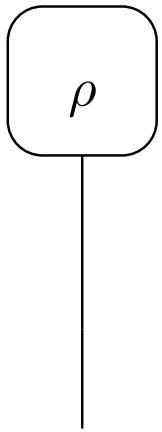

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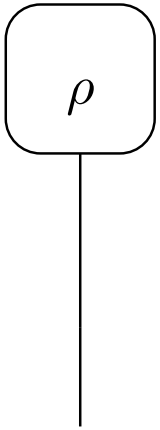
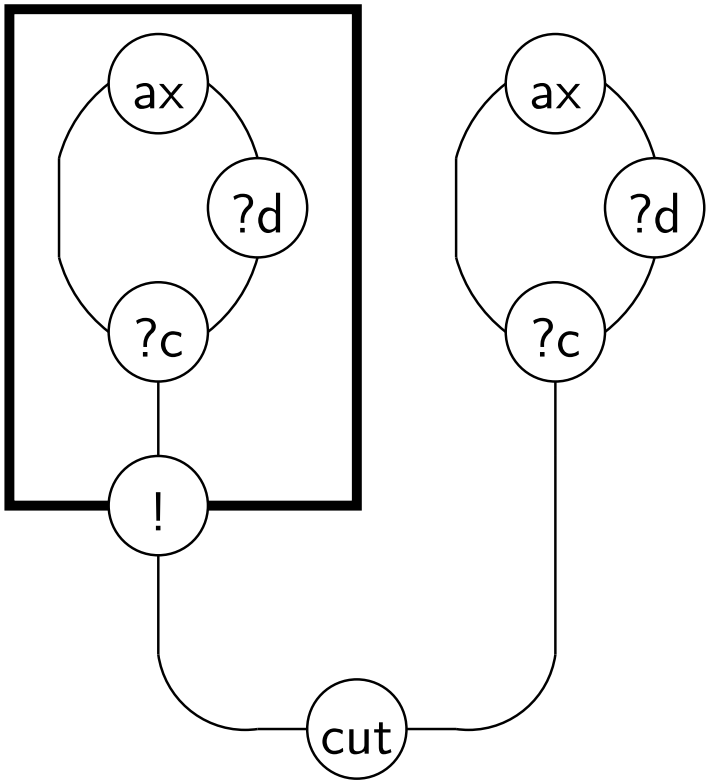


- Observation: apart from type-checking the cut, we never used types/formulas.

Untyped nets

- In fact, cut-elimination in nets makes sense even without formulas!



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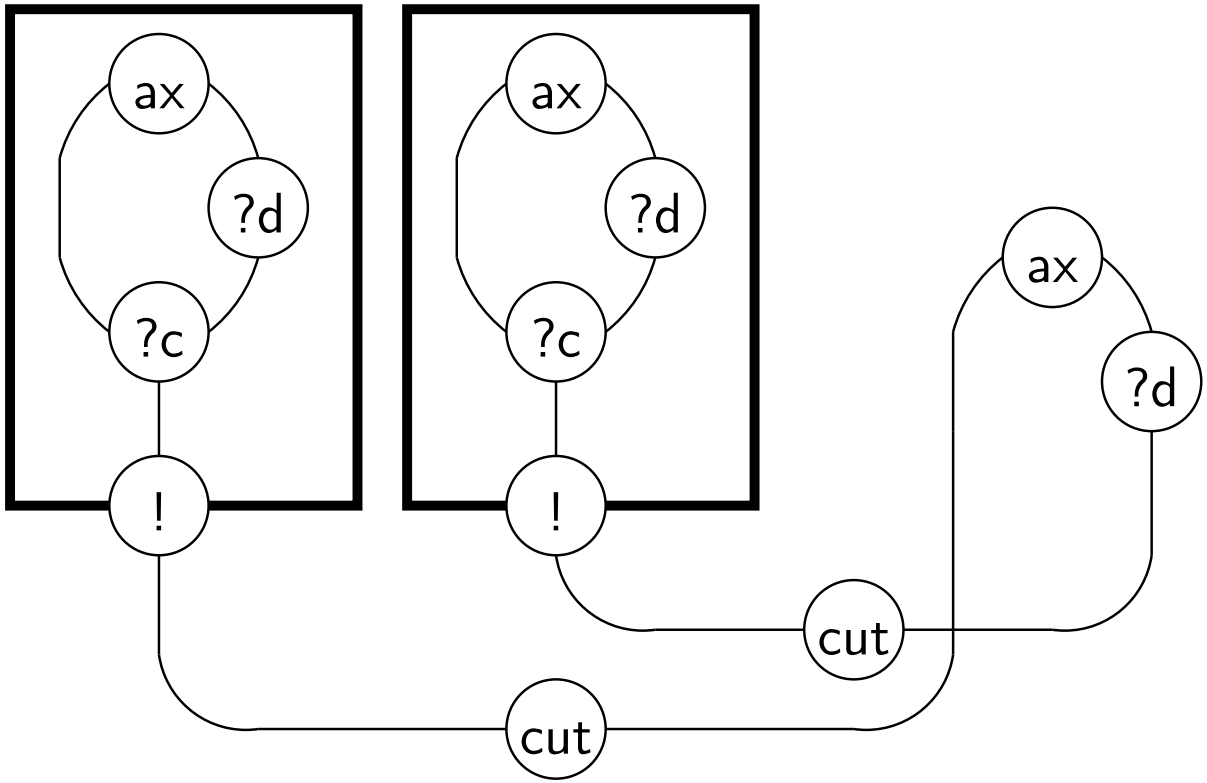

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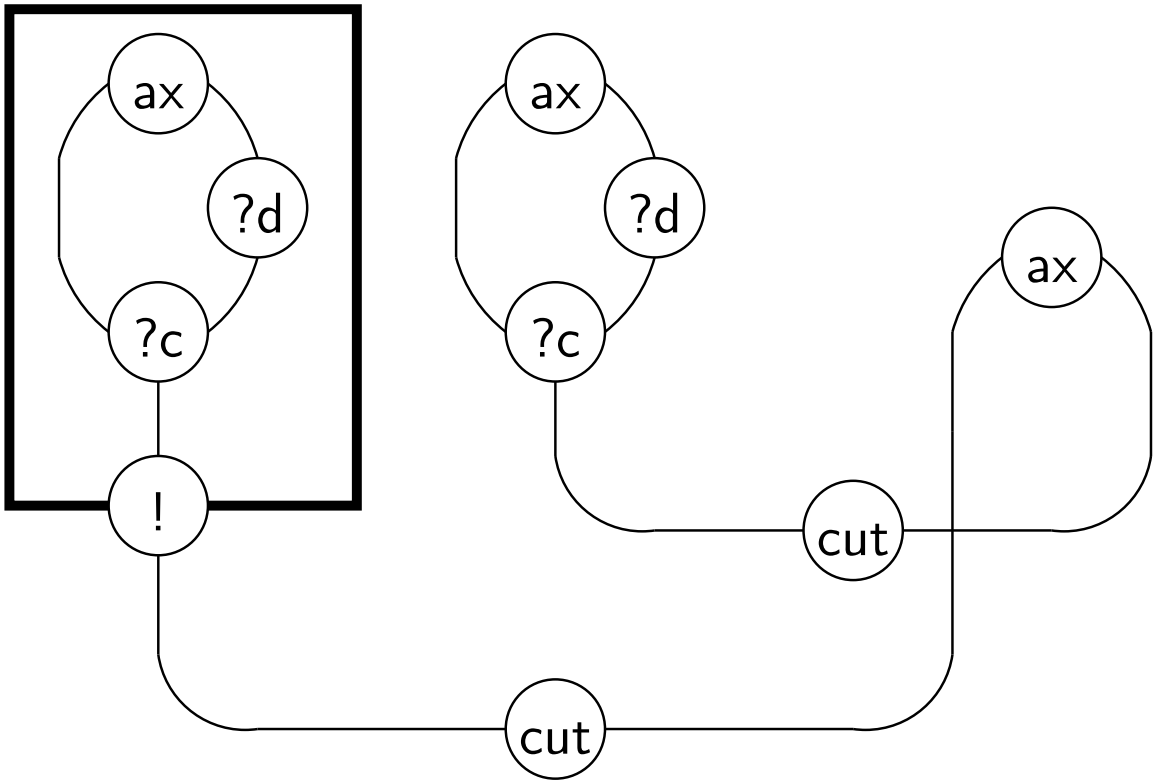

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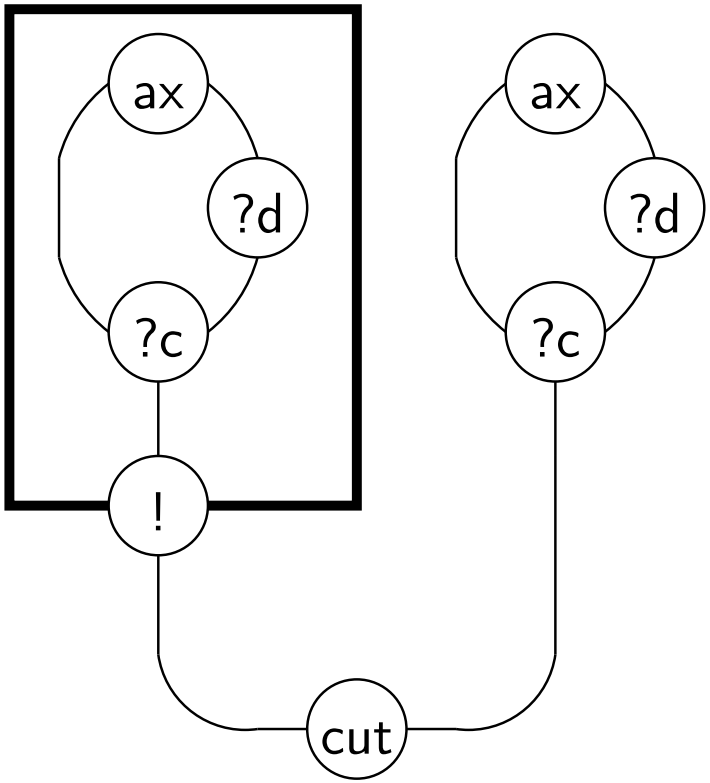

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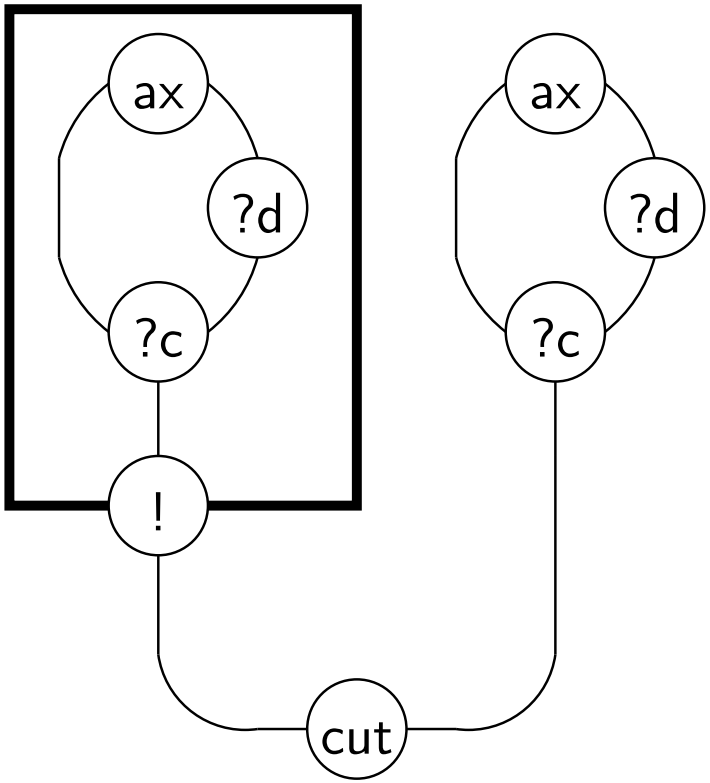

- Untyped nets are a Turing-complete model of computation.

Untyped nets

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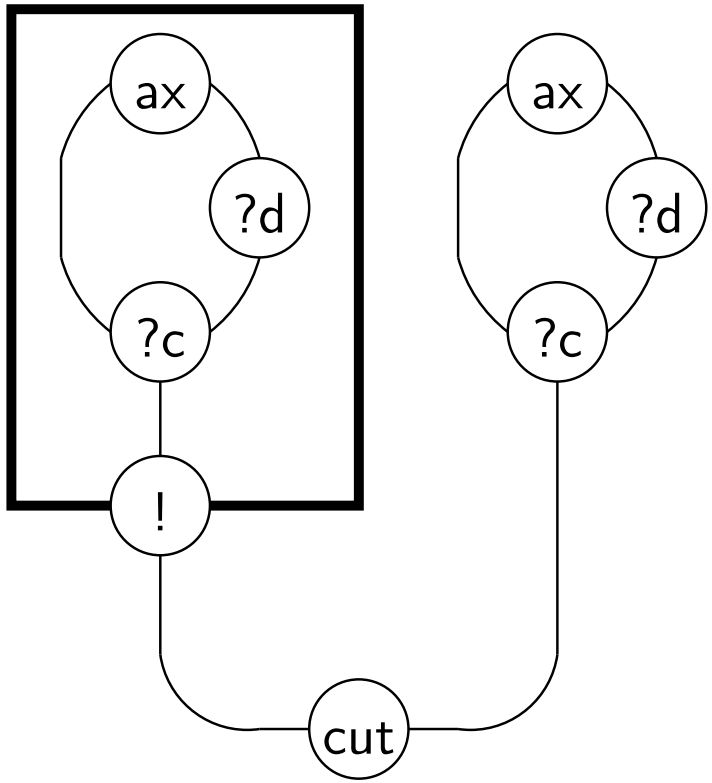
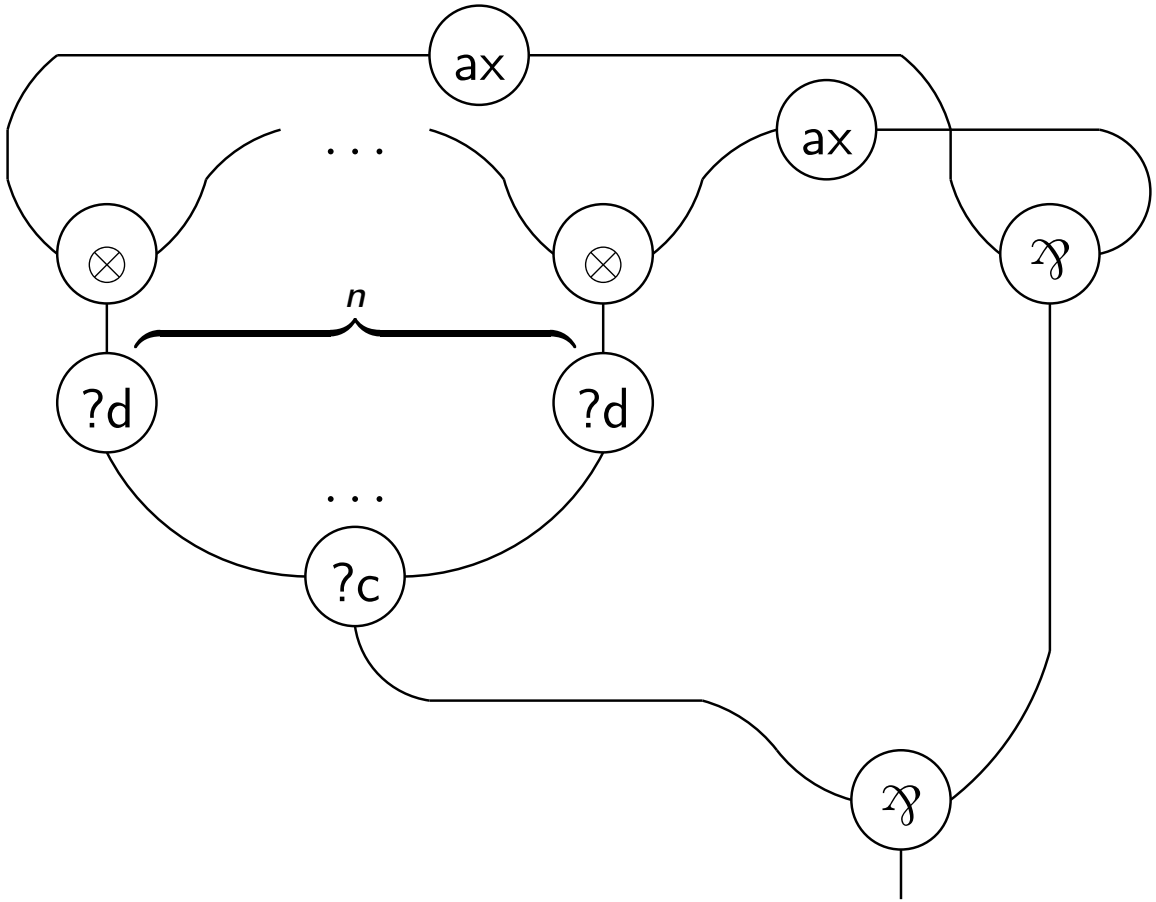


- Untyped nets are a Turing-complete model of computation.
- Actually, the above net may be typed in presence of $!R^{\perp} = R$ (Russel's antinomy).

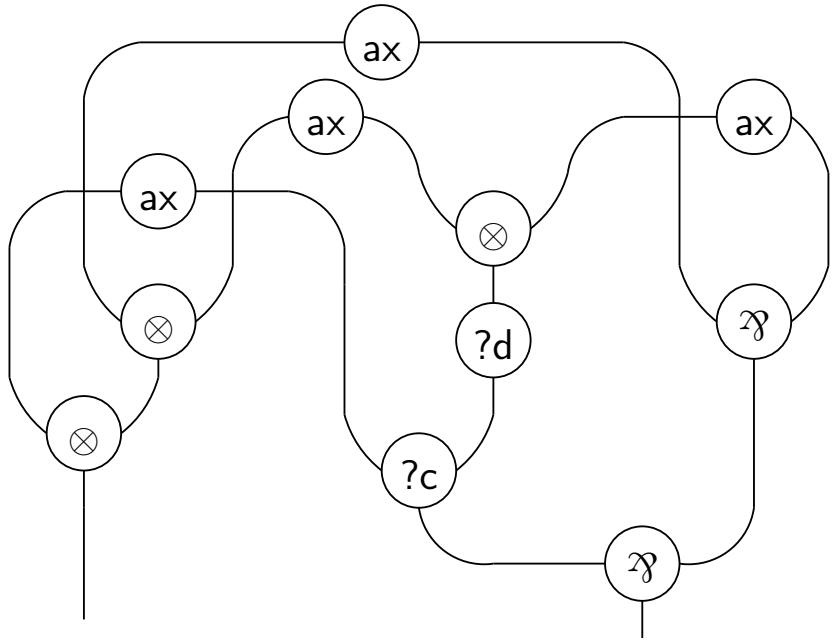
Integers

Voilà the n-th (Church) numeral, denoted by $\overline{n}$:



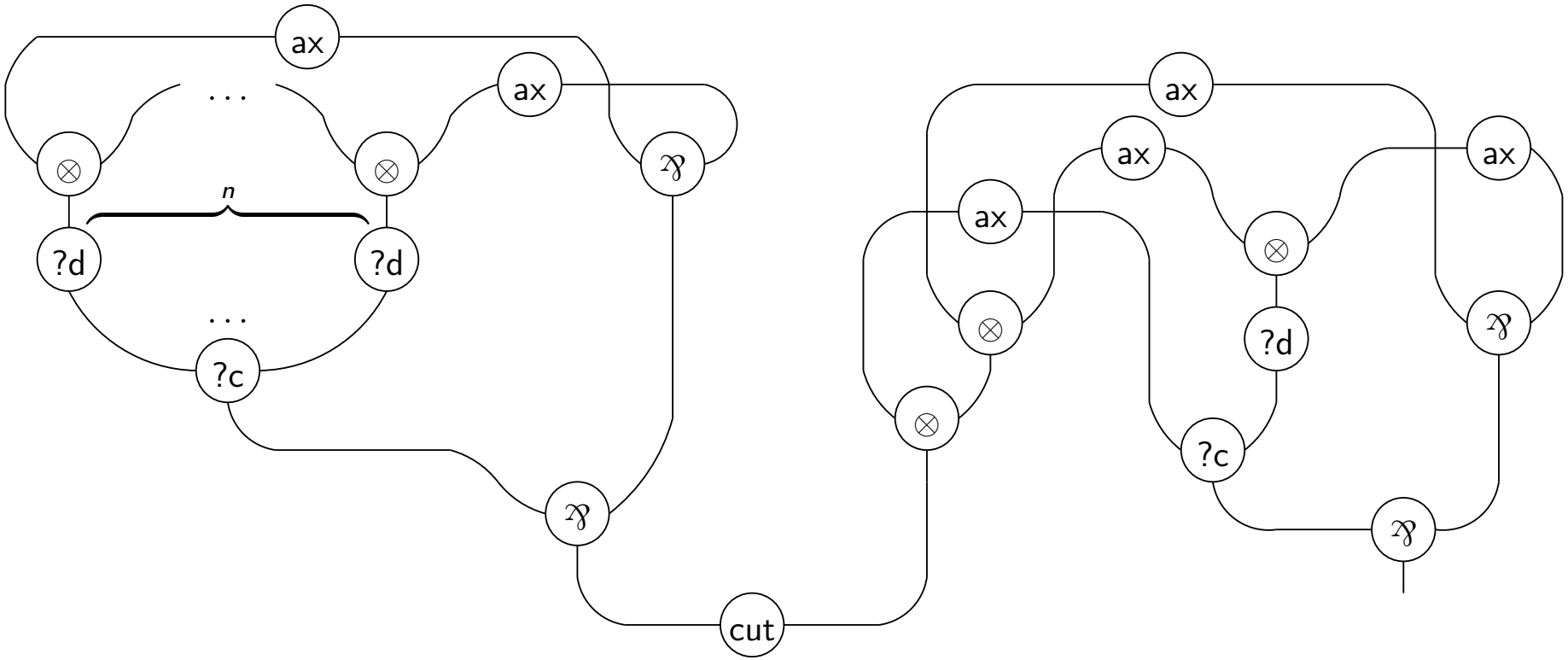
Integers

What does the following net compute?



Integers

Let's find out!



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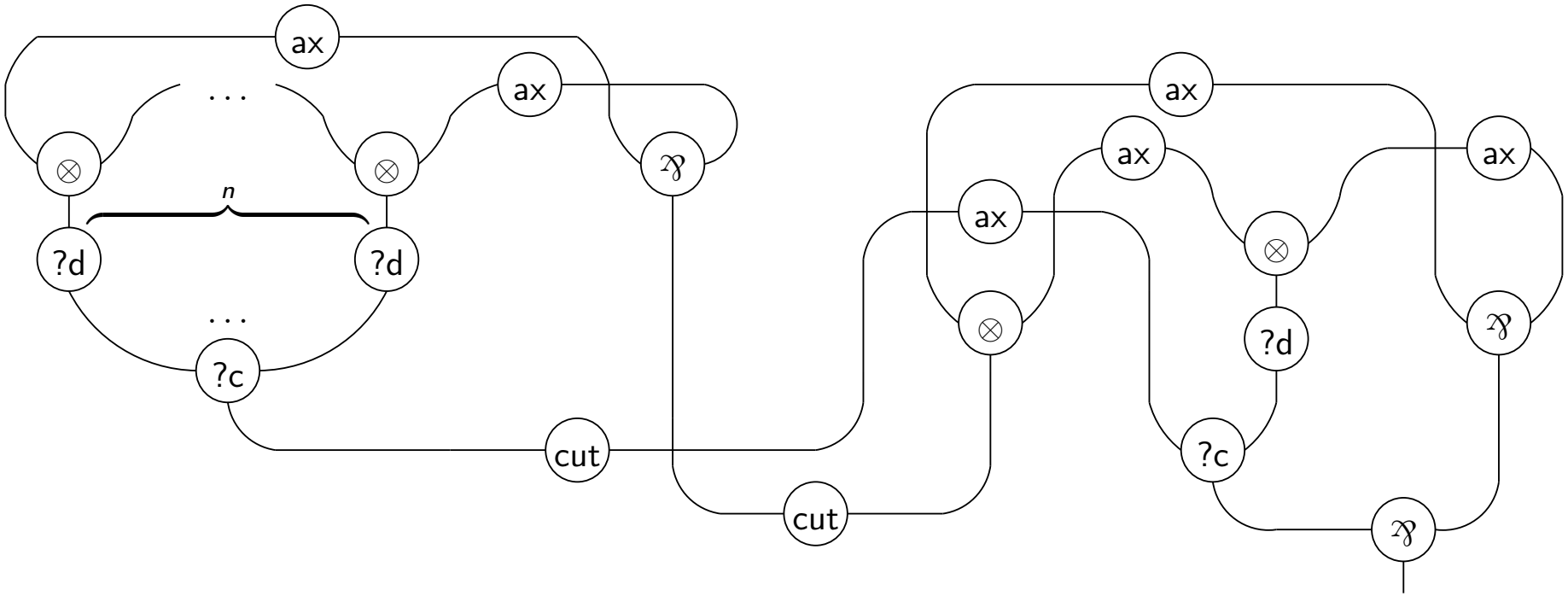

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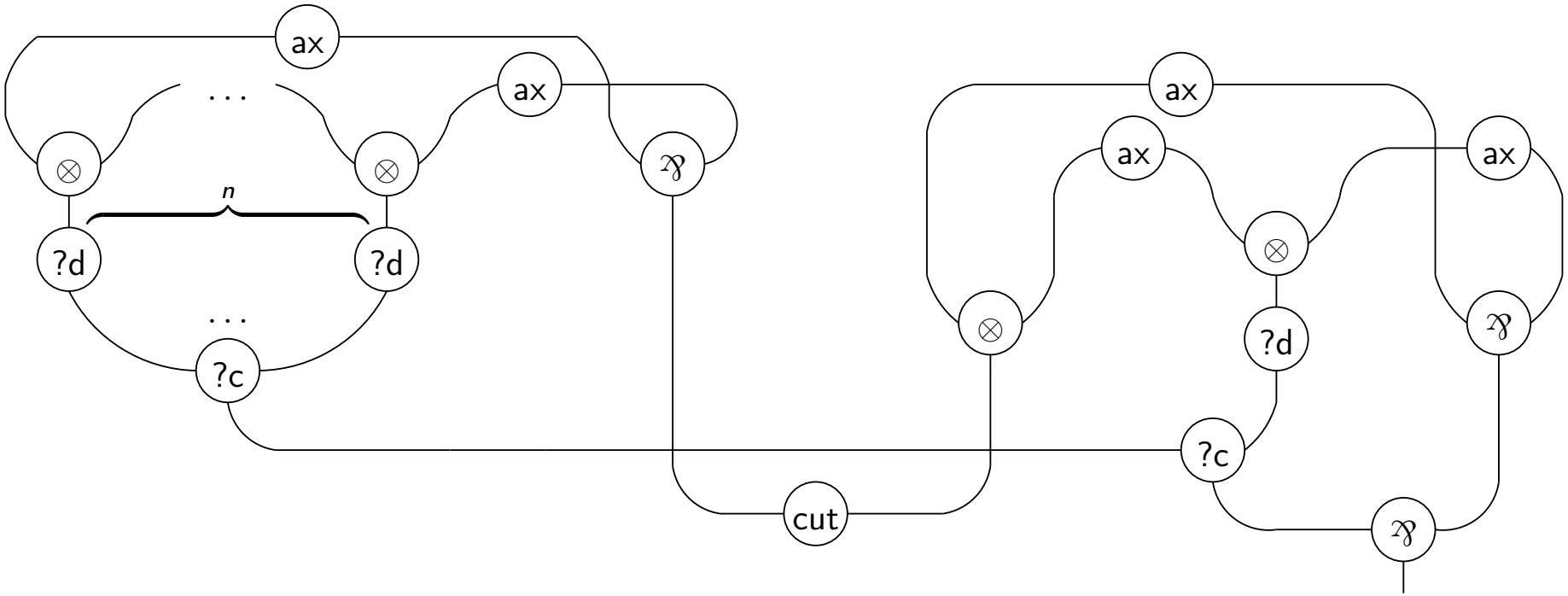

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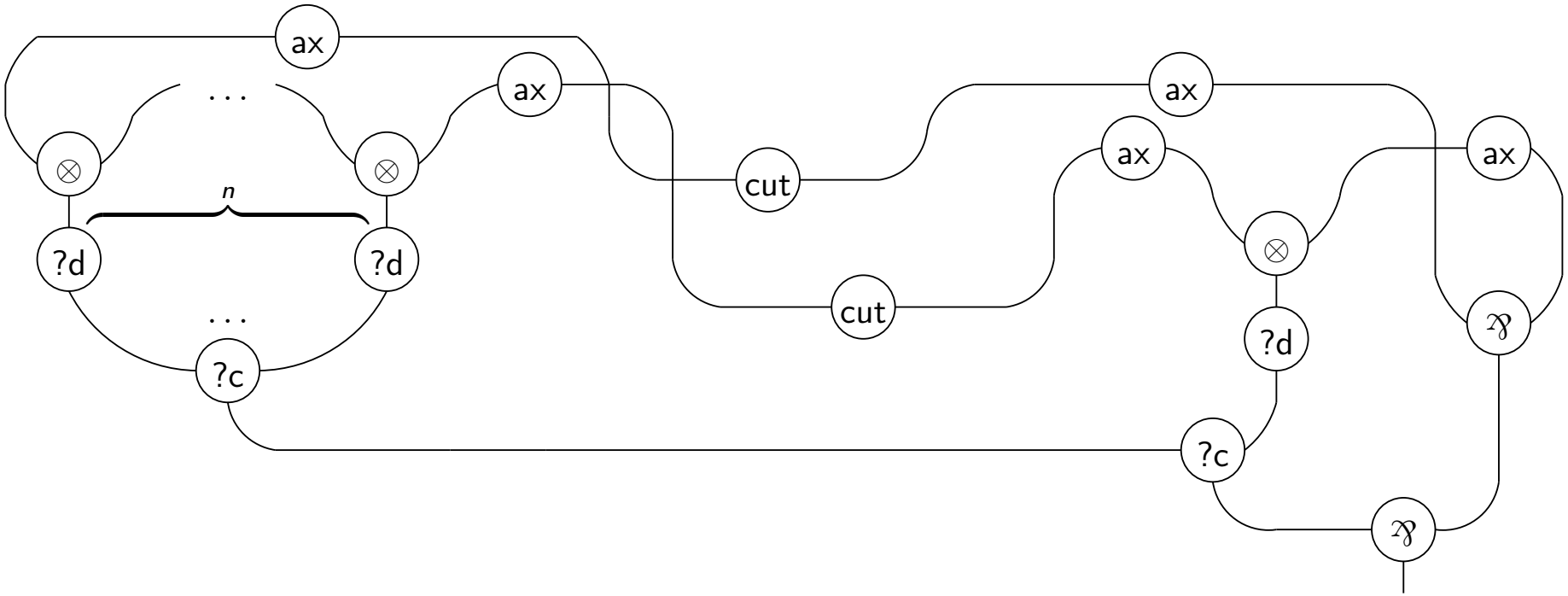

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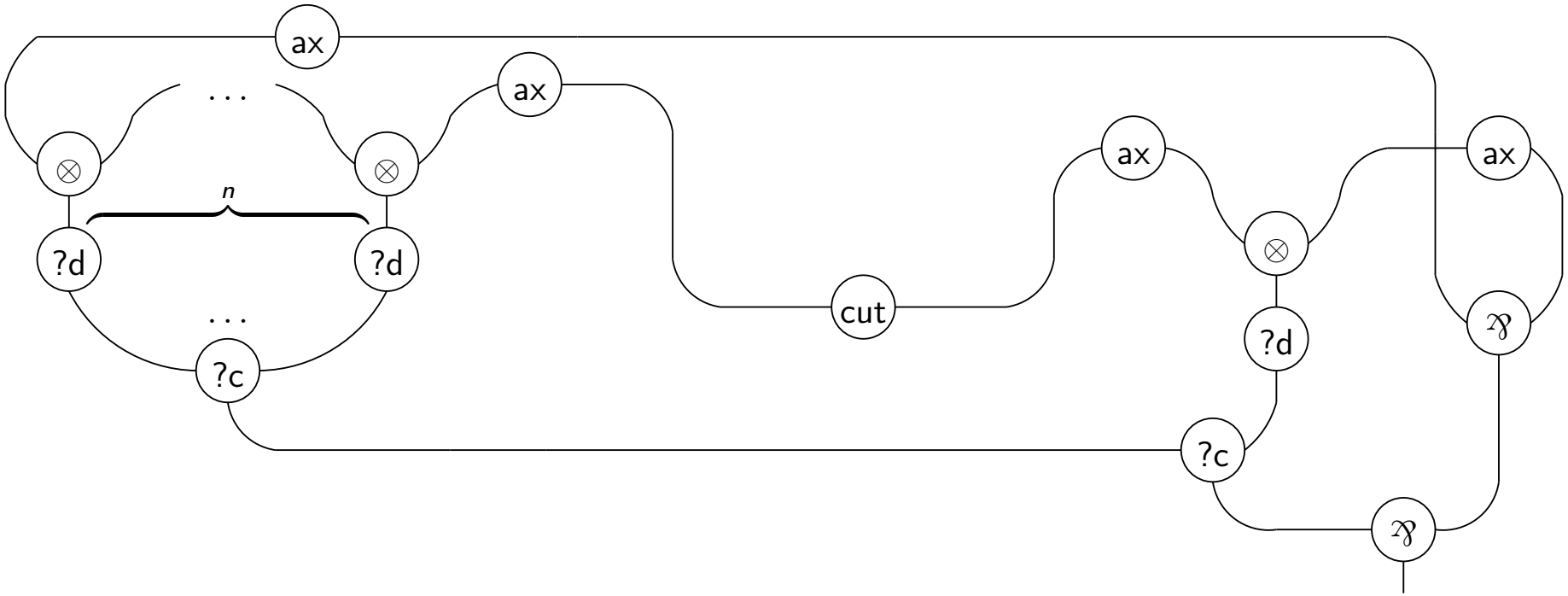

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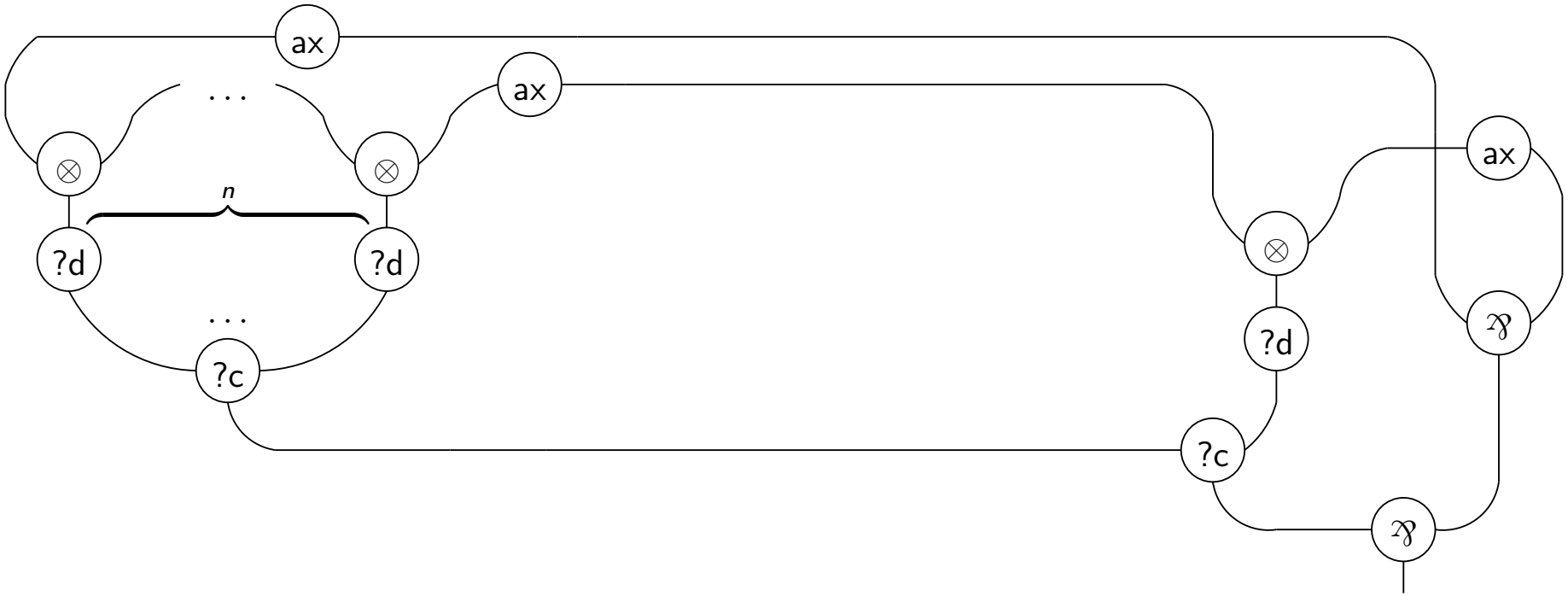
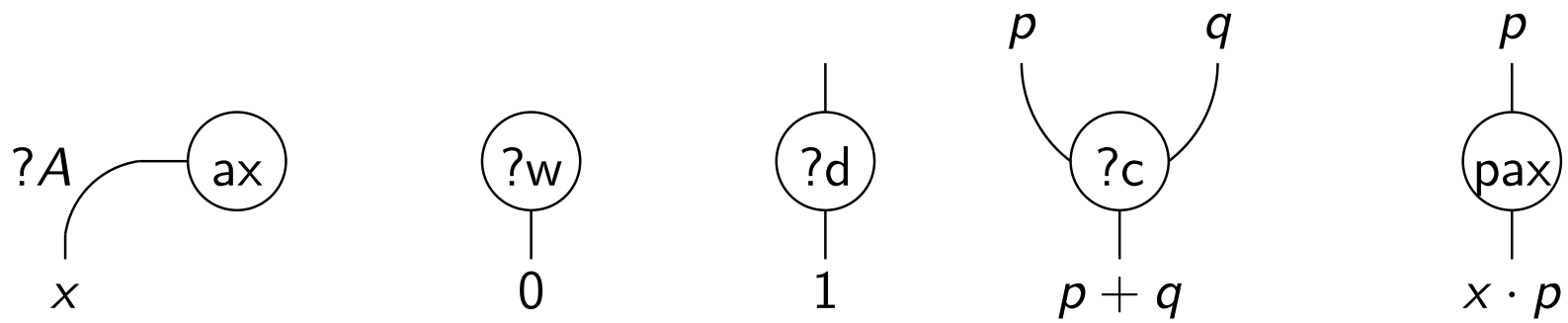


It is the successor function!

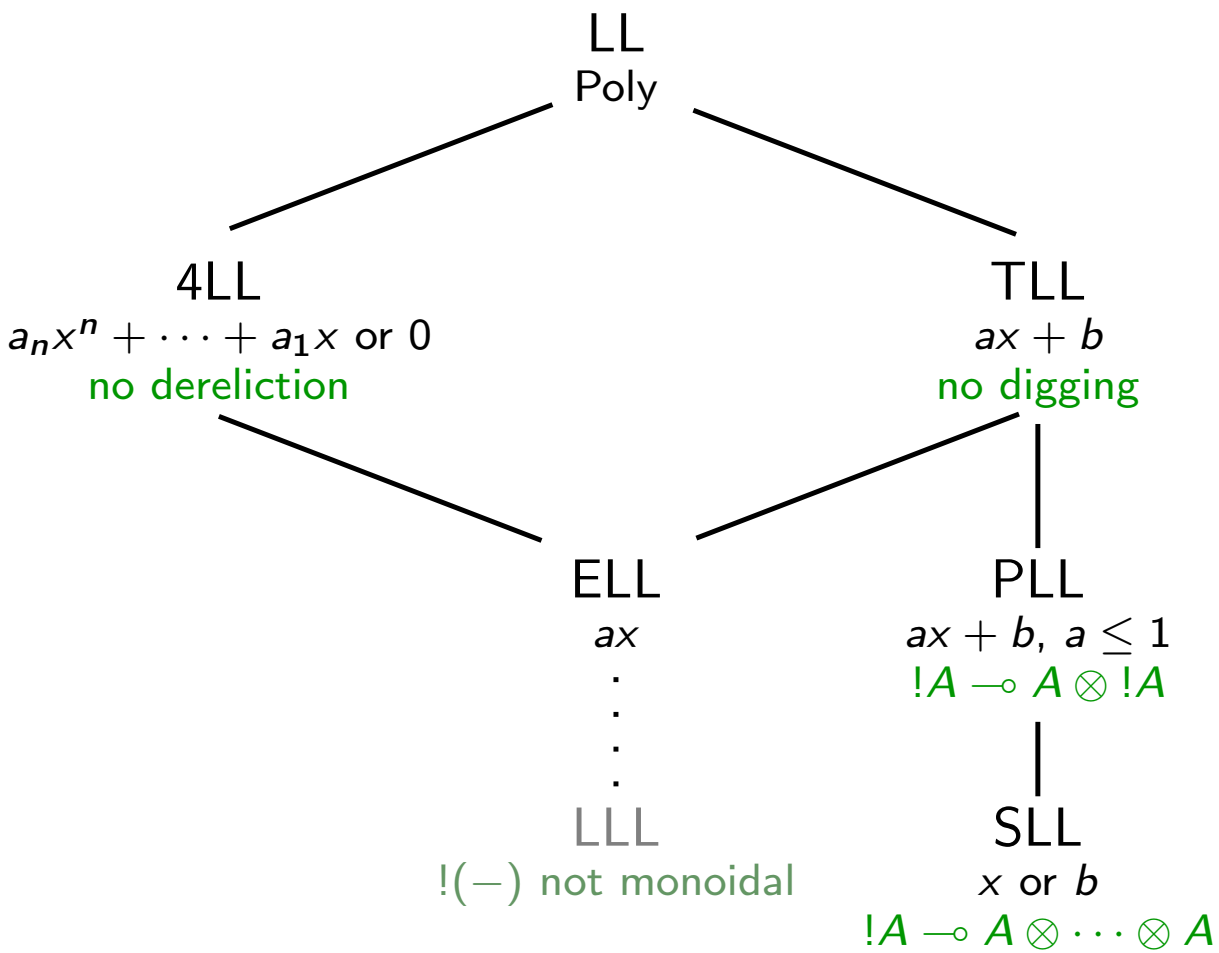
Heterodox exponentials

- Every occurrence of formula of the form $?A$ in a proof net may be assigned a polynomial with non-negative integer coefficients:



- Let Poly $:= (\mathbb{N}[x], \circ, x)$ be the monoid of polynomials under composition.
- Any submonoid $M \subseteq$ Poly induces a subsystem of linear logic (closed under cut-elimination and proving $\vdash ?A^{\perp}, !A$ for all A), as follows:
  Definition. *Call an occurrence of $?A$ in a proof net final if it is not the premise of a* ?c *or* pax *node. A proof net ρ belongs to* MELL_M *just if, whenever $?A$ is a final occurrence of ρ whose associated polynomial is p, we have $p \in M$.*

Heterodox exponentials



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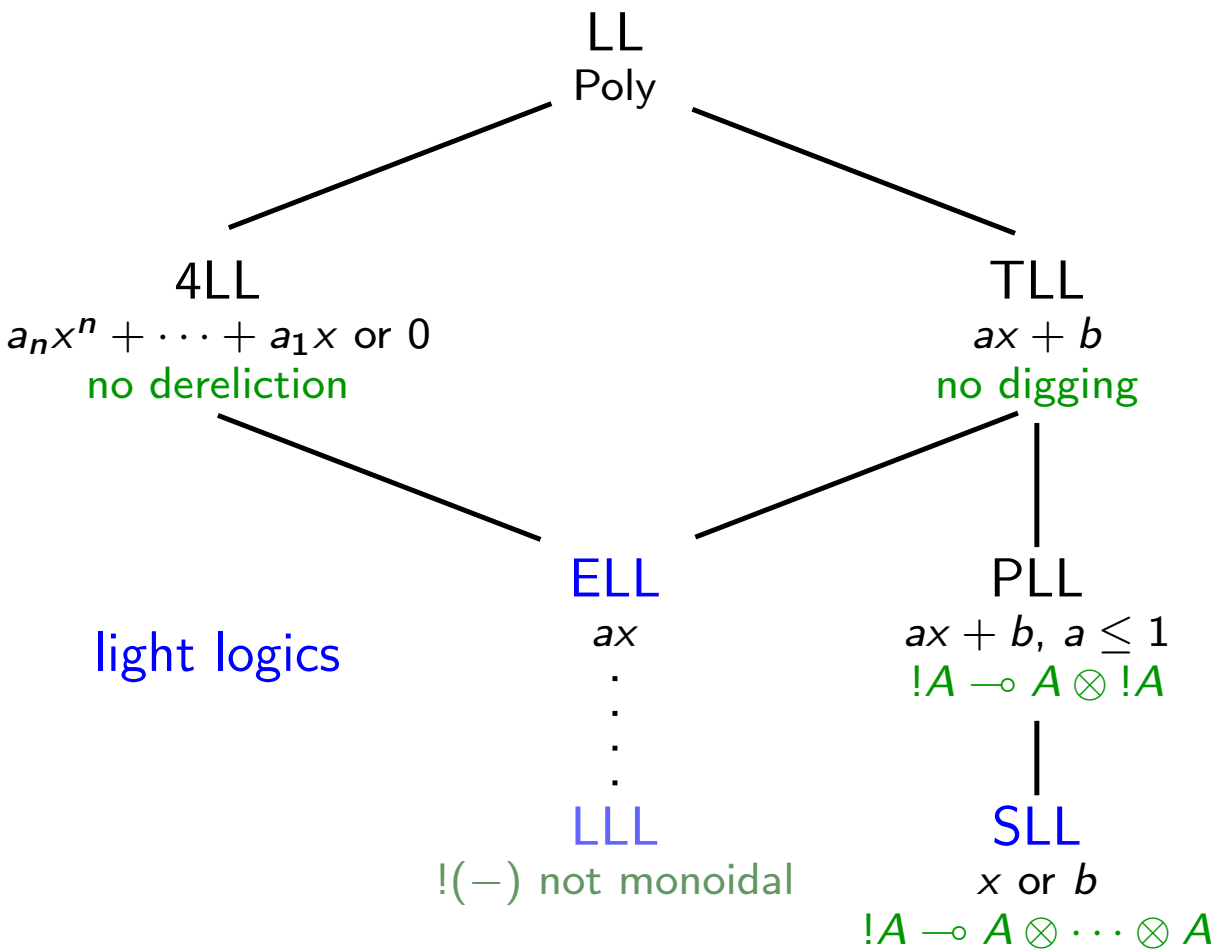

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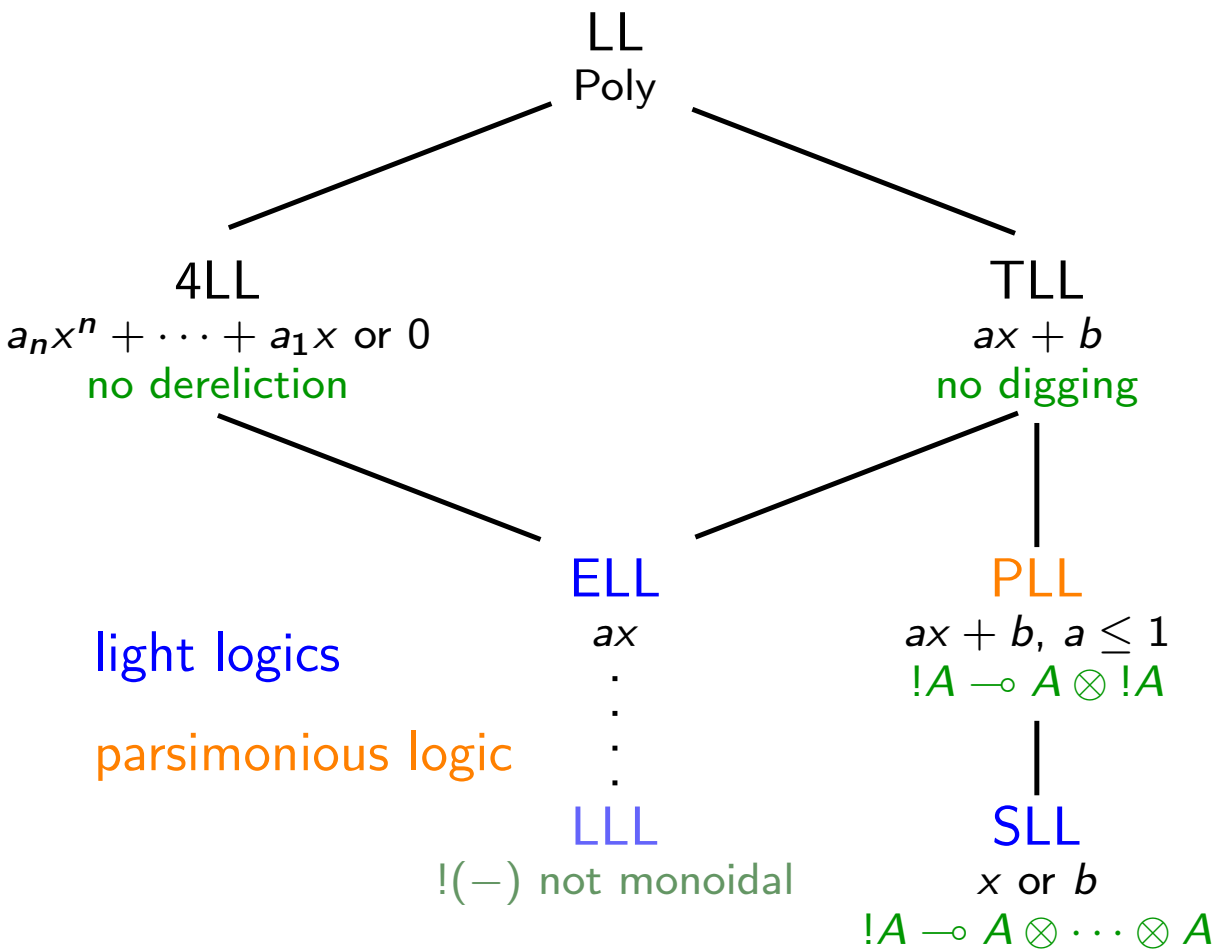

Implicit computational complexity

- Runtime = number of cut-elimination steps to normal form.
- Light logics have untyped cut-elimination:

  ELL: characterizes elementary time
  SLL: characterizes polynomial time
  LLL: characterizes polynomial time

- Parsimonious logic (with $!A \cong A \otimes !A$) is Turing-complete when untyped; however:

  propositional: characterizes logarithmic space
  linear 2nd order: characterizes polynomial time

- Two different approaches:

  stratification: (light logics) complexity is controlled globally;
  parsimony: complexity is controlled locally.

- The parsimonious approach also opens the way to non-uniformity, via *approximations*.

Perspectives

- Approximations (exponentials as a limit):
  - differential linear logic (DiLL) and Taylor expansion;
  - affine approximations and complexity.
- Quantitative analyses:
  - in DiLL, exponentials have non-deterministic cut-elimination;
  - bounded linear logic (modalities parametriezed by semi-ring).
- Geometry of interaction: an execution model based on tokens moving along proof nets.