Multiplicative Exponential Linear Logic (MELL)

Damiano Mazza
LIPN, CNRS - Université Paris 13, France

LL2016, Lyon
school: 7 and 8 November 2016

Exercise 1 (Exponential laws) *Remember the exponential axioms:*
functoriality: $!(A \multimap B) \multimap\, !A \multimap\, !B$;
dereliction: $!A \multimap A$;
digging: $!A \multimap\, !!A$;
weakening: $!A \multimap 1$;
contraction: $!A \multimap\, !A \otimes !A$.

- *Prove each axiom in sequent calculus (remember that* $A \multimap B = A^\perp \parr B$*).*
- *Translate into nets and check correctness.*

Exercise 2 (The type of integers) *The type of integers is*

$$\mathsf{Nat} \quad := \quad !(X \multimap X) \multimap X \multimap X \quad = \quad ?(X \otimes X^\perp) \parr X^\perp \parr X,$$

where X *is a fixed atom.*

- *Check that the proof nets representing the integers may be typed with* Nat.
- *Let* δ *be the following sequent calculus derivation:*

$$\begin{prooftree}
\AxiomC{}\RightLabel{ax}\UnaryInfC{$\vdash X^\perp, X$}
\AxiomC{}\RightLabel{ax}\UnaryInfC{$\vdash X^\perp, X$}
\AxiomC{}\RightLabel{ax}\UnaryInfC{$\vdash X^\perp, X$}
\RightLabel{$\otimes$}\BinaryInfC{$\vdash X^\perp, X \otimes X^\perp, X$}
\RightLabel{$\otimes$}\BinaryInfC{$\vdash X^\perp, X \otimes X^\perp, X \otimes X^\perp, X$}
\RightLabel{$\parr$}\UnaryInfC{$\vdash X \otimes X^\perp, X \otimes X^\perp, X^\perp \parr X$}
\end{prooftree}$$

  Translate it into proof nets. This net is very close to representing an integer. Which one?

- *Consider the following derivation (where* δ *is as above):*

$$\begin{prooftree}
\AxiomC{}\RightLabel{ax}\UnaryInfC{$\vdash ?(X \otimes X^\perp), !(X^\perp \parr X)$}
\AxiomC{}\RightLabel{ax}\UnaryInfC{$\vdash ?(X \otimes X^\perp), !(X^\perp \parr X)$}
\AxiomC{$\vdots\ \delta$}\noLine\UnaryInfC{$\vdash X \otimes X^\perp, X \otimes X^\perp, X^\perp \parr X$}
\RightLabel{$\otimes$}\BinaryInfC{$\vdash ?(X \otimes X^\perp), \mathsf{Nat}^\perp, X \otimes X^\perp, X^\perp \parr X$}
\RightLabel{$\otimes$}\BinaryInfC{$\vdash ?(X \otimes X^\perp), ?(X \otimes X^\perp), \mathsf{Nat}^\perp, \mathsf{Nat}^\perp, X^\perp \parr X$}
\RightLabel{$?c$}\UnaryInfC{$\vdash ?(X \otimes X^\perp), \mathsf{Nat}^\perp, \mathsf{Nat}^\perp, X^\perp \parr X$}
\RightLabel{$\parr$}\UnaryInfC{$\vdash \mathsf{Nat}^\perp, \mathsf{Nat}^\perp, \mathsf{Nat}$}
\end{prooftree}$$

  Translate it into proof nets and check that it implements addition (maybe just test it on an example).

- *Consider the following derivation (where* δ *is as above):*

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots\ \delta}{\vdash X \otimes X^\perp, X \otimes X^\perp, X^\perp \parr X}}{\vdash ?(X \otimes X^\perp), X \otimes X^\perp, X^\perp \parr X}\ ?d}{\vdash ?(X \otimes X^\perp), ?(X \otimes X^\perp), X^\perp \parr X}\ ?d}{\vdash ?(X \otimes X^\perp), ?(X \otimes X^\perp), !(X^\perp \parr X)}\ !}{\vdash ?(X \otimes X^\perp), !(X^\perp \parr X)}\ ?c \qquad \dfrac{}{\vdash X \otimes X^\perp, X^\perp \parr X}\ ax}{\dfrac{\vdash ?(X \otimes X^\perp), \mathsf{Nat}^\perp, X^\perp \parr X}{\vdash \mathsf{Nat}^\perp, \mathsf{Nat}}\ \parr}\ \otimes
$$

Translate it into proof nets and check that it implements multiplication by 2 (maybe just test it on an example).

Exercise 3 (Approximation theorem) *In the sequel, we work with atomic axioms (this does not affect provability). Let* $\mathsf{MLL}^\bullet$ *be* MLL *with two additional dual modalities, denoted by* $A^\bullet$ *and* A° *(dual means that* $(A^\bullet)^\perp = (A^\perp)^\circ$ *and vice versa), with the following sequent calculus rules:*

$$
\frac{\vdash \Gamma}{\vdash \Gamma, A^\circ} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A^\circ} \qquad \frac{\vdash \Gamma^\circ, A}{\vdash \Gamma^\circ, A^\bullet}
$$

(NB: these modalities may be defined in MALL*:* $A^\bullet := A \,\&\, 1$ *and* $A^\circ := A \oplus \bot$*).*

Let

$$
!_n A := \overbrace{A^\bullet \otimes \cdots \otimes A^\bullet}^{n}, \qquad\qquad ?_n A := \overbrace{A^\circ \parr \cdots \parr A^\circ}^{n}.
$$

- *Prove the following:*

  Theorem 1 (Approximation Theorem) *Let* $\vdash \Gamma$ *be a provable* MELL *sequent, in which each occurrence of* $!$ *(resp.* $?$*) is numbered in* $1, \ldots, m$ *(resp.* $1, \ldots, n$*). Then, there exist* n *integer polynomials* $p_1, \ldots, p_n : \mathbb{N}^m \longrightarrow \mathbb{N}$ *such that, for every* $x = (x_1, \ldots, x_m) \in \mathbb{N}^m$ *and for every* $y_1 \geq p_1(x), \ldots, y_n \geq p_n(x)$*, the sequent* $\vdash \Gamma'$ *obtained by replacing the* i*-th occurrence of* $!$ *(resp. the* j*-th occurrence of* $?$*) in* Γ *with* $!_{x_i}$ *(resp.* $?_{y_j}$*) is provable in* $\mathsf{MLL}^\bullet$*.*

  (Hint: the proof is by induction on the last rule of the derivation of Γ*, which may supposed to be cut-free. The tricky case is contraction; you may want to do directly that one).*

- *Let* ρ *be a net. A* size assignment *for* ρ *is an assignment of non-negative integers to the edges of* ρ *labelled by exponential formulas such that:*
  - *the conclusion of a dereliction is assigned* 1*;*
  - *the conclusion of a weakening is assigned* 0*;*
  - *if the premises of a contraction are assigned* p *and* q*, then its conclusion is assigned* $p + q$*;*
  - *if the premise of a* pax *link* ℓ *is assigned* p *and the conclusion of the* $!$ *node of the box of* ℓ *is assigned* r*, then the conclusion of* ℓ *is assigned* $r \cdot p$*;*
  - *if the premises of a cut are assigned* p *and* q*, then* $p = q$*.*

  Show that size assignments are preserved by exponential cut-elimination: if $\rho \longrightarrow \rho'$ *by means of an exponential cut-elimination step and* ρ *has a size assignment* s*, then* ρ' *has a size assignment* s' *which coincides with* s *on every edge whose labelling formula appears in the conclusions.*

- *Verify (even just informally) that the assignment of integers to occurrences of modalities given by the Approximation Theorem induces a size assignment for the corresponding proof net. Deduce that there is no proof of* $\vdash \mathsf{Nat}^\perp, \mathsf{Nat}$ *in* MELL *implementing the exponential function.*