Introductory school in Linear Logic

INTRODUCTION

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LL2016, Lyon
school: 7 and 8 November 2016
2016: the words ‘Linear Logic’ cover a huge field

- Various chapters and tools, among which:
  - Sequent calculi
  - Proofs-nets
  - Denotational semantics
  - Game semantics and Ludics
  - Geometry of interaction
  - Implicit computational complexity
  - Categorical approaches
  - etc

- Various proofs systems
  - fragments, variants, extensions of the original LL: e.g. MLL, ALL, MALL, MELL, LL, ELL, LLL etc
  - Linear Logics
1986: birth of LL

- Due to Jean-Yves Girard, a french logician
- LL borns at the meeting point of two scientific lines:
  - *Proof theory* (Logic)
  - *Computing theory* (Denotational semantics of programs)

What denoted the words “Linear Logic” at the very beginning?

- A sequent calculus logical system
- “Yet another formal logic”? No.
- Linear Logic = Classical logic, but decomposed and observed through the microscope of the computational point of view on proofs
- Goal of this introduction: make understand what this means...
Make understand what it means that:

“Linear Logic is just **Classical logic** but **decomposed** through the **microscope** of the computational point of view on proofs”

**PLAN:**

- PART 1. Proofs in classical logic (static)
- PART 2. The computational point of view on proofs (dynamic)
- PART 3. Classical logic decomposed
PART 1
PROOFS IN CLASSICAL LOGIC
Important steps in the history of Proofs representation

David Hilbert’s Proof theory program

▶ Prove the consistency of formal methods (Peano axioms for arithmetics)

▶ Through a new branch of mathematics: “Proof theory”, studying mathematically mathematical proofs
Important steps in the history of Proofs representation

David Hilbert’s Proof theory program

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- Through a new branch of mathematics: “Proof theory”, studying mathematically mathematical proofs
- What is a proof?
Important steps in the history of Proofs representation

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- Through a new branch of mathematics: “Proof theory”, studying mathematically mathematical proofs
- What is a proof?
  - Hilbertian answer: a discourse respecting the rules of logic (local correctness)
David Hilbert’s Proof theory program

▶ Prove the consistency of formal methods (Peano axioms for arithmetics)

▶ Through a new branch of mathematics: “Proof theory”, studying mathematically mathematical proofs

▶ What is a proof?
  ▶ Hilbertian answer: a discourse respecting the rules of logic (local correctness)
  ▶ Other possible answers: e.g. a universal strategy against any argumentative attack (dialectic answer)
Important steps in the history of Proofs representation

- Proofs as “texts" (written in the ideographical language of formal logic)
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Important steps in the history of Proofs representation

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  - Natural Deduction, G. Gentzen, +/-1930

Let us start with Natural Deduction:
Important steps in the history of Proofs representation

- Proofs as “texts” (written in the ideographical language of formal logic)
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  - Sequent calculus

Let us start with Natural Deduction:

Proofs as texts, deriving statements (formulas) from statements (formulas)
Important steps in the history of Proofs representation

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- As non textual objects: JY. Girard’s Proof-nets
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  - Spatial 3-dimensional objects
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  - non sequential
  - global correctness
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  - non sequential
  - global correctness

Let us start with Natural Deduction :

- Proofs as texts,
- deriving statements (formulas) from statements (formulas)
Propositional formulas

Example (a formula):

\[((\neg X \to Y) \land X) \lor \neg(Y \to \bot)\]

Formulas $A, B, C \ldots$ are inductively built:

- from elementary formulas:
  - atoms: $X, Y, Z$ etc (atomic formulas)
  - and absurdum: $\bot$

- by applying:
  - the unary constructor negation: $\neg A$
  - the binary constructors:
    - conjunction: $A \land B$
    - disjunction: $A \lor B$
    - implication: $A \to B$
Natural Deduction: General shape of proofs (tree)
Natural Deduction: General shape of proofs (tree)

\[\begin{align*}
&\text{Hypothesis} \\
&\downarrow \\
&\text{Formula} \\
&\downarrow \\
&\text{Formula} \\
&\frac{\text{Formula}}{\text{Rule}} \ldots \\
&\frac{\text{Formula}}{\text{Rule}} \\
&\frac{\text{Formula}}{\text{Rule}} \\
&\vdots \\
&\frac{\text{Formula}}{\text{Rule}} \\
&\vdots \\
&\frac{\text{Formula}}{\text{Rule}} \\
&\uparrow \\
&\text{Conclusion}
\end{align*}\]
Natural Deduction: General shape of proofs (tree)

Hypothesis
\[ \downarrow \]
Formula
\[ \downarrow \]
Formula
\[ \overline{rule} \]
Formula
\[ \overline{rule} \]
Formula
\[ \overline{rule} \]
Formula
\[ \overline{rule} \]
Formula
\[ \overline{rule} \]
Formula
\[ \overline{rule} \]
Formula
\[ \overline{rule} \]
Formula
\[ \overline{rule} \]
Formula

Hypothesis 1 \[ \ldots \] Hypothesis k

Conclusion
Proofs in Natural Deduction: an example

\[
\frac{X \to (Y \to Z)}{Y \to Z} \quad X \to \text{élim} \\
\frac{X \to Y}{Y \to Z} \quad X \to \text{élim}
\]

Hypothesis 1 \ldots Hypothesis k

Conclusion
Proofs in Natural Deduction: an example

X → (Y → Z)

Y → Z

X →-élim

X

Y → Z

→-élim

X

Y

→-élim

A multiset of formulas

\[ \{ X \rightarrow (Y \rightarrow Z), X \rightarrow Y, Y \rightarrow Z \} \]

Hypothesis 1 \ldots Hypothesis k

Conclusion

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Introductory school in Linear Logic

INTRO
Proofs in Natural Deduction: an example

A multiset of formulas

\[ X \rightarrow (Y \rightarrow Z) \]
\[ \frac{X}{Y \rightarrow Z} \frac{X}{-\text{elim}} \]
\[ \frac{X \rightarrow Y}{Z} \frac{X}{-\text{elim}} \]

Hypothesis 1 \ldots Hypothesis k

Conclusion
Proofs in Natural Deduction: an example

\[ \frac{X \rightarrow (Y \rightarrow Z)}{} \]
\[ \frac{X}{Y \rightarrow Z} \rightarrow \text{elim} \]
\[ \frac{X \rightarrow Y}{Z} \rightarrow \text{elim} \]

A multiset of formulas

Hypothesis 1 \ldots \text{Hypothesis } k

Conclusion

A multiset = a set "with repetitions" (no matter the order)
Notations:

- Formulas: $A, B, C$ etc
- Multisets of formulas: $\Gamma, \Delta$ etc
- Proofs: $\pi$ etc

Simply represented by:

$\Gamma \vdash \pi \Gamma \vdash \pi \Delta \vdash \pi A$
### Rules

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<th>Introductions</th>
<th>Eliminations</th>
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<td>$\frac{A \quad B}{A \land B} \quad \land\text{-intro}$</td>
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<td>$\frac{A \land B}{B} \quad \land\text{-elim (2)}$</td>
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<td>$\frac{A \lor B}{A \lor B} \quad \lor\text{-intro (1)}$</td>
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Remark. Rules are constructors for proofs (not transitions from formulas to formulas)

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<td>$\frac{A \rightarrow B \land \text{intro}}{ \frac{A}{B} \land \text{intro (2)}}, \quad \frac{B}{A \lor B} \lor\text{-intro (2)}</td>
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#### Remark

Rules are constructors for proofs (not transitions from formulas to formulas)
Logical rules actually are proofs constructors

For instance, the rule $\land$-elim$_1$ presented as:

\[
\begin{array}{c}
\Gamma \\
\vdots \\
A \land B \\
\hline
A \\
\end{array}
\]

$\land$-elim$_1$
Logical rules actually are proofs constructors

For instance, the rule $\land$-elim$_1$ presented as:

\[
\begin{array}{c}
\Gamma \\
\vdots \\
A \land B \\
\end{array} \quad \frac{A \land B}{A} \quad \land$-elim$_1
\]

is just a shortcut for:

\[
\begin{array}{c}
\Gamma \\
\vdots \\
\pi \\
A \land B \\
\end{array} \quad \rightarrow \\
\begin{array}{c}
\Gamma \\
\vdots \\
\pi \\
A \land B \\
\end{array} \quad \frac{A \land B}{A} \quad \land$-elim$_1
\]

Proof $\pi$
constructed at Time $t$

New proof $\pi'$
constructed at Time $t + 1$
### Rules

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**Remark 3. Hypothesis desactivation**

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\[ \frac{n}{\frac{B}{A \rightarrow B}} \rightarrow \text{-intro} \]

Remark 3. **Hypothesis desactivation**
Hypothesis
“active”
at Time \( t \) . . .

\[
\Gamma \\
\begin{array}{c}
A \ldots A \\
\vdots \\
B \\
\end{array} \\
: \pi \\
\]

Proof \( \pi \) 
constructed at Time \( t \)

\ldots are 
desactivated
at Time \( t + 1 \)

\[
\Gamma \\
\begin{array}{c}
A \ldots A \\
\vdots \\
\end{array} \\
: \pi \\
\]

\[
\begin{array}{c}
k \text{ times} \\
\end{array} \\
\Gamma \\
\begin{array}{c}
A \ldots A \\
\vdots \\
B \\
\end{array} \\
\begin{array}{c}
\rightarrow \text{-intro} \\
\end{array} \\
\]

\[
\begin{array}{c}
k \text{ times} \\
\end{array} \\
\Gamma \\
\begin{array}{c}
A \ldots A \\
\vdots \\
A \rightarrow B \\
\end{array} \\
\begin{array}{c}
\rightarrow \text{-intro} \\
\end{array} \\
\]

New proof \( \pi' \) 
constructed at Time \( t + 1 \)
For any formula \( A \), this one node tree below is a proof:

\[ \begin{array}{c}
A
\end{array} \]

In that tree, the formula \( A \) is both:

- a leaf: so it is a proof under hypothesis \( A \)
- the root: so it is a proof with conclusion \( A \)

So is a proof of \( A \) under hypothesis \( A \).

Terminology: “identity axiom”
For any formula $A$, this one node tree below is a proof:

$A$
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- a leaf: so it is a proof under hypothesis $A$
- the root: so it is a proof with conclusion $A$

So is a proof of $A$ under hypothesis $A$.

Terminology: “identity axiom”
A proof in Natural Deduction: an example

\[
\begin{align*}
1 &\quad X & \rightarrow\text{-elim} \\
2 &\quad Y & \rightarrow\text{-elim} \\
3 &\quad Z & \rightarrow\text{-intro} \\
4 &\quad (X \rightarrow Y) & \rightarrow\text{-intro} \\
5 &\quad (X \rightarrow (Y \rightarrow Z)) & \rightarrow\text{-intro} \\
\end{align*}
\]
PART 2

The computational point of view on proofs
An example of non analytical proof

\[ \begin{array}{c}
\frac{1}{X} \\
\frac{Y}{X \land Y} & \text{\texttt{\&-intro}} \\
\frac{X}{X} & \text{\texttt{\&-elim (1)}} \\
\frac{Y \rightarrow X}{X} & \text{\texttt{\rightarrow-intro}} \\
\frac{Y \rightarrow X}{X \rightarrow (Y \rightarrow X)} & \text{\texttt{\rightarrow-intro}} \\
\end{array} \]
An example of non analytical proof

The connective $\land$ is not present in $X \rightarrow (Y \rightarrow X)$ (the proved theorem)
An example of non analytical proof

The connective $\land$ is not present in $X \rightarrow (Y \rightarrow X)$ (the proved theorem)

- $\land$ is an extrinsic element: it cannot be obtained by analyzing the theorem (nor the hypothesis)
An example of non analytical proof

The connective \( \land \) is not present in \( X \rightarrow (Y \rightarrow X) \) (the proved theorem)

\( \land \) is an extrinsic element: it cannot be obtained by analyzing the theorem (nor the hypothesis)

The proof is **not analytical** (it contains an extrinsic element)
An example of non analytical proof

\[
\begin{array}{c}
\frac{1}{X} & \frac{2}{Y} \\
& \hline
& \frac{X \land Y}{X} \quad \land\text{-intro} \\
& \frac{X}{X} \quad \land\text{-elim (1)} \\
& \frac{Y \rightarrow X}{Y \rightarrow X} \quad \rightarrow\text{-intro} \\
& \frac{X \rightarrow (Y \rightarrow X)}{X \rightarrow (Y \rightarrow X)} \quad \rightarrow\text{-intro}
\end{array}
\]

- The connective $\land$ is not present in $X \rightarrow (Y \rightarrow X)$ (the proved theorem)
- $\land$ is an extrinsic element: it cannot be obtained by analyzing the theorem (nor the hypothesis)
- The proof is not analytical (it contains an extrinsic element)

**Importance of analyticity, w.r.t. heuristics (proof-search):** knowing the statement we want to prove, we know in advance the finite list of formulas that could appear in the proof we are looking for (namely the subformulas of the statement supposed to be proved).
Back to our example of non analytical proof

\[
\begin{array}{c}
\frac{1}{2} \quad \frac{1}{1} \quad X \\
\frac{X \land Y}{X} \quad ^\land\text{-intro} \\
\frac{X \land Y}{X} \quad ^\land\text{-elim (1)} \\
\frac{1}{Y \rightarrow X} \quad ^\rightarrow\text{-intro} \\
\frac{2}{X \rightarrow (Y \rightarrow X)} \quad ^\rightarrow\text{-intro}
\end{array}
\]
Back to our example of non analytical proof

\[
\begin{align*}
2 & \quad \text{X} \\
1 & \quad \text{Y} \\
\hline
\quad & \quad \text{X} \land \text{Y} \\
\quad & \quad \text{X} \\
1 & \quad \text{Y} \rightarrow \text{X} \\
2 & \quad \text{X} \rightarrow (\text{Y} \rightarrow \text{X})
\end{align*}
\]

\begin{itemize}
\item \text{X} \land \text{Y} \quad \text{^intro}
\item \text{X} \quad \text{^elim (1)}
\item \text{Y} \rightarrow \text{X} \quad \rightarrow\text{intro}
\item \text{X} \rightarrow (\text{Y} \rightarrow \text{X}) \quad \rightarrow\text{intro}
\end{itemize}

▶ In that proof : scarcely introduced, the extrinsic element \text{^} happens to be eliminated
Back to our example of non analytical proof

\[
\begin{array}{c}
\frac{X}{X \land Y} \\
\frac{X}{X} \\
\frac{Y \rightarrow X}{X \rightarrow (Y \rightarrow X)}
\end{array}
\quad \left\{ \begin{array}{l}
\text{\textbf{\^{^\wedge}}-intro} \\
\text{\textbf{\^{^\wedge}}-elim (1)} \\
\text{\rightarrow-intro}
\end{array} \right. \\
\text{a “cut” (alias a “redex”)}
\]

▶ In that proof: scarcely introduced, the extrinsic element \(\land\) happens to be eliminated
Back to our example of non analytical proof

▶ In that proof: scarcely introduced, the extrinsic element $\land$ happens to be eliminated

▶ Gentzen proved:
  ▶ The situation above is general: **Proofs with no cut are analytical** (NB: wrong in second order logic)
Back to our example of non analytical proof

\[
\begin{array}{c}
1 \quad Y \\
2 \quad X \\
\hline
X \land Y \\
\hline
X \\
\hline
Y \rightarrow X \\
\hline
X \rightarrow (Y \rightarrow X)
\end{array}
\]

- \^intro
- \^elim (1)
- \^elim (1)
- \rightarrow intro
- \rightarrow intro

- In that proof: scarcely introduced, the extrinsic element \( \land \) happens to be eliminated

- Gentzen proved:
  - The situation above is general: **Proofs with no cut are analytical** (NB: wrong in second order logic)
  
  - **Analytizability of proofs**: one can transform any proof, in a cut-free (hence analytical) proof of the same theorem.
Gentzen’s algorithm for cut elimination: one step

\[
\begin{align*}
\Gamma_1 & \quad A^{\langle 0 \rangle} \ldots A^{\langle k \rangle} \\
& \quad \vdash \pi_1 \\
\Gamma_2 & \quad B \\
\Gamma & \quad \Gamma_2 \langle 0 \rangle \ldots \pi_2 \langle k \rangle \\
B & \quad \vdash \pi_1 \\
A & \quad \vdash \pi_2 \\
A \rightarrow B & \quad \rightarrow \text{-intro} \\
B & \quad \rightarrow \text{-elim}
\end{align*}
\]
Gentzen’s algorithm for cut elimination: one step

\[ \Gamma_1 \vdash A^{\langle 0 \rangle} \ldots A^{\langle k \rangle} \]

\[ n \vdash B \quad A \rightarrow B \quad \rightarrow - \text{intro} \]

\[ A \quad \rightarrow - \text{elim} \]

\[ \Gamma_2 \]

\[ \vdash \pi_1 \]

\[ \vdash \pi_2 \]

\[ \Rightarrow \]

\[ \Gamma_1 \vdash A^{\langle 0 \rangle} \ldots A^{\langle n \rangle} \]

\[ \vdash \pi_1 [\pi_2 / A] \]

\[ B \]

\[ \vdash \]
Gentzen’s algorithm for cut elimination: one step

\[ \Gamma_1 \quad \langle 0 \rangle \quad ... \quad \langle k \rangle \]

\[
\begin{array}{c}
\Gamma_2 \\
\pi_1 \\
\pi_2 \\
\end{array}
\]

\[\frac{\Gamma_1 \quad \langle 0 \rangle \quad ... \quad \langle n \rangle}{\frac{\langle 0 \rangle}{A}} \quad \rightarrow\text{-intro} \quad \frac{\langle k \rangle}{B} \quad \rightarrow\text{-elim} \quad \frac{\langle 0 \rangle}{A} \quad ... \quad \frac{\langle n \rangle}{A}
\]

\[\Rightarrow \quad \Gamma_2 \quad \langle 0 \rangle \quad ... \quad \langle k \rangle \]

\[\Gamma_1 \quad \langle 0 \rangle \quad ... \quad \langle n \rangle \]

\[\pi_1 \quad \pi_2 \]

\[\pi_1 \left[ \pi_2 / A \right] \]

\[B \]

\[\vdots \]

\[\vdots \]

- Complexity ⇔ Duplication ⇔ Multiple occurrences hypothesis
Why is cut-eliminability so important?

▶ Logical reasons: Non analytical proofs

▶ Logical reasons: Non intrinsic motivation of the theorem

▶ Logical reasons: No bound on the number of different formulas to consider in the heuristical (proof-search) process

A corollary of cut-eliminability: Consistency results

For Proof theory (and in particular Linear Logic), cut-elimination is the cornerstone of Logic.

▶ Computing theory reasons:

▶ Lambda calculus (Church, 1934): A functional ontology; a programming language

▶ Curry-Howard isomorphism (1969): (Typed Lambda calculus, execution) = (Natural Deduction, cut-elimination)

Cut-elimination is a conceptual bridge between Logic and Computing theory.
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    - =
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Cut-elimination is a conceptual bridge between Logic and Computing theory.
### Terminology and notations for Natural Deduction systems:

- **Minimal**: $NM$
- **Intuitionistic**: $NJ = NM + efq$
- **Classical**: $NK = NJ + raa$
The “symmetries of classical logic” : dual connectives

- Notation : $A \equiv B$ if $A$ and $B$ are provably equivalent in $NK$
- De Morgan “laws” :

$$\neg\neg A \equiv A$$

$$\neg(A \land B) \equiv \neg A \lor \neg B \quad \neg A \land \neg B \equiv \neg(A \lor B)$$
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- “Symmetries of classical logic” at the provability level
- However those symmetries:
  - are not visible in $NK$ at the level of proofs
  - i.e. are almost not reflected in $NK$ rules
### The “dissymmetries of classical Natural Deduction”

<table>
<thead>
<tr>
<th>Introductions</th>
<th>Eliminations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[ \frac{A \land B}{A} \And\text{-elim (1)} ] [ \frac{A \land B}{B} \And\text{-elim (2)} ]</td>
</tr>
<tr>
<td>[ \frac{A}{A \lor B} \lor\text{-intro (1)} ] [ \frac{B}{A \lor B} \lor\text{-intro (2)} ]</td>
<td></td>
</tr>
<tr>
<td><strong>Intuitionistic absurdum (negation)</strong></td>
<td><strong>Classical absurdum (negation)</strong></td>
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- **Remark Symmetry between introductions for** \( \lor \) **and elimination for** \( \land \) (**“De Morgan laws”**).
The “dissymmetries of classical Natural Deduction”

<table>
<thead>
<tr>
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</table>
| \[
\frac{A}{A \land B} \quad \frac{B}{A \land B} \quad \land\text{-intro}
\] | \[
\frac{A \land B}{A} \quad \land\text{-elim (1)} \quad \frac{A \land B}{B} \quad \land\text{-elim (2)}
\] |
| \[
\frac{A}{A \lor B} \quad \lor\text{-intro (1)} \quad \frac{B}{A \lor B} \quad \lor\text{-intro (2)}
\] | \[
\frac{}{\Gamma} \quad \left( \frac{}{\pi_1} \quad \frac{A}{\Gamma} \right) \quad \left( \frac{}{\pi_2} \quad \frac{B}{\Gamma} \right) \quad \lor\text{-elim}
\] |
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<table>
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| \[
\frac{A}{A \wedge B} \quad \frac{B}{A \wedge B}
\] \wedge\text{-intro} & \[
\frac{A \wedge B}{A} \quad \frac{A \wedge B}{B}
\] \wedge\text{-elim (1)} \quad \wedge\text{-elim (2)}

\[
\frac{A}{A \lor B} \quad \frac{B}{A \lor B}
\] \lor\text{-intro (1)} \quad \lor\text{-intro (2)} & \[
\frac{\pi_1}{\Gamma \ \ A} \quad \frac{\pi_2}{\Gamma \ \ B}
\] 
\[
\frac{\pi_2}{\ n \quad A \vee B \quad C}
\] \lor\text{-elim}

\[
\frac{\perp}{\neg A}
\] \neg\text{-intro} & \[
\frac{\neg A}{\Gamma}
\] \neg\text{-elim}

\[
\frac{\perp}{A}
\] \text{efq} & \[
\frac{\Gamma \quad n \neg A}{A}
\] \text{raa}

### Remark
Symmetry between introductions for \(\lor\) and elimination for \(\wedge\) ("De Morgan laws").
Two dissymmetries in the Natural Deduction format:

- Dissymmetry Hypothesis/Conclusion (*many hypothesis vs one conclusion*)
- Proofs are “conclusion oriented” (the grow down: toward conclusion)
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Two reasons to move from Natural Deduction to Sequent calculus
- Recover classical symmetries at the level of proofs (rules)
- Get rid of difficulties when generalizing cut elimination to classical Natural Deduction
An arithmetical calculation represented via moves from arithmetical expressions to other ones:

\[(3 \times 2) \times (5 \times 3)\]

= 

\[6 \times (5 \times 3)\]

= 

\[6 \times 15\]

= 

\[90\]

Representation of the same calculation via moves from arithmetical identities to other ones:

\[(3 \times 2) \times (5 \times 3) = (3 \times 2) \times (5 \times 3)\]

\[\Downarrow\]

\[(3 \times 2) \times (5 \times 3) = 6 \times (5 \times 3)\]

\[\Downarrow\]

\[(3 \times 2) \times (5 \times 3) = 6 \times 15\]

\[\Downarrow\]

\[(3 \times 2) \times (5 \times 3) = 90\]
Natural Deduction deriving sequents from sequents

Representation of the progression of a proof via steps from formulas to formulas:

\[
\begin{align*}
  X \land ((Y \land Z) \land W) & \quad \Rightarrow & \quad (Y \land Z) \land W \\
  (Y \land Z) \land W & \quad \Rightarrow & \quad Y \land Z \\
  Y \land Z & \quad \Rightarrow & \quad (Y \land Z) \lor X
\end{align*}
\]

Representation of the same proof via steps from ‘sequents’ to ‘sequents’:

\[
\begin{align*}
  X \land ((Y \land Z) \land W) & \quad \Rightarrow & \quad X \land ((Y \land Z) \land W) \\
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  X \land ((Y \land Z) \land W) & \quad \Rightarrow & \quad (Y \land Z) \lor X
\end{align*}
\]

Two different notations for the same proof
Natural Deduction deriving sequents from sequents

Representation of the progression of a proof via steps from formulas to formulas:

\[
\begin{align*}
  X \land ((Y \land Z) \land W) & \vdash (Y \land Z) \land W & \land\text{-elim}_1 \\
  (Y \land Z) \land W & \vdash Y \land Z & \land\text{-elim}_2 \\
  Y \land Z & \vdash (Y \land Z) \lor X & \lor\text{-intro}_1 \\
\end{align*}
\]

Representation of the same proof via steps from ‘sequents’ to ‘sequents’:

\[
\begin{align*}
  X \land ((Y \land Z) \land W) & \vdash X \land ((Y \land Z) \land W) & \land\text{-elim}_1 \\
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  X \land ((Y \land Z) \land W) & \vdash (Y \land Z) \lor X & \lor\text{-intro}_1 \\
\end{align*}
\]

Two different \textit{notations} for the same proof
Deduction of formulas versus Derivation of sequents

A deduction of formulas from formulas

\[ \text{Hypothesis 1 \ldots Hypothesis k} \]

\[ \text{Conclusion} \]

A derivation of sequents from sequents

\[ \text{Hypothesis 1, \ldots, Hypothesis k} \vdash \text{Conclusion} \]
Deduction of formulas versus Derivation of sequents

A deduction of formulas from formulas

Hypothesis 1 \[\ldots\] Hypothesis k

\[\Gamma_1 A \ldots A\]

\[\vdash B\]

\[\frac{\Gamma, A, \ldots, A}{\Gamma \vdash A \rightarrow B}\]

→-intro

Conclusion

A derivation of sequents from sequents

Hypothesis 1, \ldots, Hypothesis k \[\vdash\] Conclusion

A sequent

\[\vdash B\]

\[\frac{\Gamma, A, \ldots, A}{\Gamma \vdash A \rightarrow B}\]

→-intro
A deduction of formulas from formulas

\[
\begin{align*}
&\text{Hypothesis } 1 \quad \ldots \quad \text{Hypothesis } k \\
&\Gamma_1, A, \ldots, A \\
&\vdash B \quad \to \text{-intro}
\end{align*}
\]

A derivation of sequents from sequents

\[
\begin{align*}
&\text{Hypothesis } 1, \ldots, \text{Hypothesis } k \vdash \text{Conclusion} \\
&\Gamma, A, \ldots, A \vdash B \\
&\Gamma \vdash A \to B \quad \to \text{-intro}
\end{align*}
\]
Deduction of formulas versus Derivation of sequents

A deduction of formulas from formulas

\[ \text{Hypothesis} \; 1 \; \ldots \; \text{Hypothesis} \; k \]

\[ \Gamma_1 \; A \; \ldots \; A \]

\[ \vdash \pi \]

\[ n \]

\[ B \; \rightarrow \text{-intro} \]

\[ A \; \rightarrow B \]

A derivation of sequents from sequents

\[ \text{Hypothesis} \; 1, \; \ldots, \; \text{Hypothesis} \; k \vdash \text{Conclusion} \]

\[ \Gamma, A, \ldots, A \vdash B \]

\[ \Gamma \vdash A \rightarrow B \; \rightarrow\text{-intro} \]
A deduction of formulas from formulas

\[ \text{Hypothesis 1 \ldots Hypothesis } k \]

\[ \text{Conclusion} \]

A derivation of sequents from sequents

\[ \text{Hypothesis 1, \ldots, Hypothesis } k \vdash \text{Conclusion} \]

\[ \text{A sequent} \]

\[ \text{Hyp} \]
\[ \downarrow \]
\[ A \quad \text{(Id ax)} \]
\[ \uparrow \]
\[ \text{Concl} \]

\[ A \vdash A \quad \text{Id Axm} \]
Classical Natural Deduction

- **Axiomes identité**

  \[
  \Gamma, A \vdash A
  \]

- **Implication**

  \[
  \Gamma, A, \cdots, A \vdash B \\
  \hline
  \Gamma \vdash A \rightarrow B
  \]

- **Négation**

  \[
  \Gamma, A, \cdots, A \vdash \bot \\
  \hline
  \Gamma \vdash \neg A
  \]

- **Conjonction**

  \[
  \Gamma \vdash A \\
  \Gamma \vdash B \\
  \hline
  \Gamma \vdash A \land B
  \]

- **Disjonction**

  \[
  \Gamma \vdash A \\
  \hline
  \Gamma \vdash A \lor B
  \]

  \[
  \Gamma \vdash B \\
  \hline
  \Gamma \vdash A \lor B
  \]

- **L’absurde**

  \[
  \Gamma \vdash \bot \\
  \hline
  \Gamma \vdash A
  \]
Classical Natural Deduction : version 1

- **Axiomes identité**
  \[ \Gamma, A \vdash A \]

- **Implication**
  \[ \Gamma, A, \ldots, A \vdash B \]
  \[ \Gamma \vdash A \rightarrow B \rightarrow \text{intro} \]

- **Négation**
  \[ \Gamma, A, \ldots, A \vdash \bot \]
  \[ \Gamma \vdash \lnot A \lnot \text{-intro} \]

- **Conjonction**
  \[ \Gamma \vdash A \]
  \[ \Gamma \vdash B \]
  \[ \Gamma \vdash A \land B \land \text{-intro} \]

- **Disjonction**
  \[ \Gamma \vdash A \]
  \[ \Gamma \vdash B \]
  \[ \Gamma \vdash A \lor B \lor \text{-intro} \]

- **L’absurde**
  \[ \Gamma \vdash \bot \]
  \[ \Gamma \vdash A \text{ efq} \]
  \[ \Gamma, \lnot A, \ldots, \lnot A \vdash \bot \lnot \text{-elim} \]
  \[ \Gamma \vdash A \raa \]
Classical Natural Deduction: version 2

- **Axiomes identité**
  \[ \Gamma, A \vdash A, \Delta \]

- **Implication**
  \[ \Gamma, A, \ldots, A \vdash B, \Delta \]
  \[ \Gamma \vdash A \rightarrow B, \Delta \rightarrow\text{-intro} \]

- **Négation**
  \[ \Gamma, A, \ldots, A \vdash \bot, \Delta \]
  \[ \Gamma \vdash \neg A, \Delta \neg\text{-intro} \]

- **Conjonction**
  \[ \Gamma \vdash A, \Delta \]
  \[ \Gamma \vdash B, \Delta \]
  \[ \Gamma \vdash A \land B, \Delta ^\land\text{-intro} \]

- **Disjonction**
  \[ \Gamma \vdash A, \Delta \]
  \[ \Gamma \vdash B, \Delta \]
  \[ \Gamma \vdash A \lor B, \Delta ^\lor\text{-i1} \]
  \[ \Gamma \vdash A \lor B, \Delta ^\lor\text{-i2} \]

- **L’absurde**
  \[ \Gamma \vdash \bot, \Delta \]
  \[ \Gamma \vdash A, \Delta \text{ efq} \]
Classical Natural Deduction: version 2

- **Axiomes identité**
  \[ \frac{\text{axm-id}}{\Gamma, A \vdash A, \Delta} \]

  Sequents \( \Gamma \vdash \Delta \) (multi-conclusions, symmetrical)

- **Implication**
  \[ \frac{\Gamma, A, \cdots, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta} \rightarrow\text{-intro} \]

- **Négation**
  \[ \frac{\Gamma, A, \cdots, A \vdash \bot, \Delta}{\Gamma \vdash \neg A, \Delta} \neg\text{-intro} \]

- **Conjonction**
  \[ \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \land B, \Delta} \land\text{-intro} \]

- **Disjonction**
  \[ \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \lor B, \Delta} \lor\text{-i1} \]
  \[ \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \lor B, \Delta} \lor\text{-i2} \]

- **L’absurde**
  \[ \frac{\Gamma \vdash \bot, \Delta}{\Gamma \vdash A, \Delta} \text{efq} \]
The first dissymmetry (Hypothesis/Conclusions) disappeared:
Sequents $\Gamma \vdash \Delta$ are symmetrical
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Not the second dissymmetry:
Proofs continue to be “conclusion directed”
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Not the second dissymmetry:
Proofs continue to be “conclusion directed”

Sequent calculus: replacing elimination on the right by Introduction on the left
Symmetries of *classical* theorems (De Morgan) now implemented in rules.
Main qualities of Sequent Calculus

- Symmetries of *classical* theorems (De Morgan) now implemented in rules

- Clear architectonic of rules. They are divided in three groups:
  - Identity Group
  - Logical Group
  - Structural Group
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  - Logical Group
  - Structural Group

- The distinction between analytic / non analytic proofs is:
  - evident
  - conceptually clear: composition of proofs

- Uncover the dynamical sense of duality: Reversible / Irreversible logical rules
# Identity Group

**Identity axiom**

\[
\text{ax} \quad \frac{}{A \vdash A}
\]

**Cut rule**

\[
\frac{\Gamma_1 \vdash \Delta_1, A \quad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \quad \text{cut}
\]
### Identity group

**Identity axiom**

\[ \text{ax} \quad \frac{}{A \vdash A} \]

**Cut rule**

\[ \frac{\Gamma_1 \vdash \Delta_1, A \quad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \text{ cut} \]

### Structural group

**Contractions**

\[ \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \text{ ctr} \]

\[ \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ ctr} \]

**Weakenings**

\[ \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \text{ w} \]

\[ \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ w} \]
### Identity group

**Identity axiom**

\[
\text{ax} \quad \frac{}{A \vdash A}
\]

### Structural group

<table>
<thead>
<tr>
<th>Contractions</th>
<th>Weaknesses</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma \vdash A, A, \Delta )</td>
<td>( \Gamma \vdash A, \Delta )</td>
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Logical group : unary connectives (negation)

<table>
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<tr>
<th>Rule</th>
<th>Negation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma, A \vdash \Delta )</td>
<td>( \Gamma \vdash \neg A, \Delta )</td>
</tr>
<tr>
<td>( \neg \Gamma \vdash \neg A, \Delta )</td>
<td>( \neg \Gamma \vdash \neg A, \Delta )</td>
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</tr>
<tr>
<td>( \neg \Gamma, \neg A \vdash \Delta )</td>
</tr>
</tbody>
</table>

### Logical group: binary dual connectives

#### Rules for conjunction

- \( \Gamma \vdash A, \Delta \)
- \( \Gamma' \vdash B, \Delta' \)
- \( \Gamma, \Gamma' \vdash A \land B, \Delta, \Delta' \)
- \( \Gamma, A \land B \vdash \Delta \)

#### Rules for disjunction

- \( \Gamma \vdash A, B, \Delta \)
- \( \Gamma \vdash A \lor B, \Delta \)
- \( \Gamma, A \lor B \vdash \Delta \)
- \( \Gamma, \Gamma', A \lor B \vdash \Delta, \Delta' \)
### Logical group: unary connectives (negation)

<table>
<thead>
<tr>
<th>Negation</th>
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</thead>
<tbody>
<tr>
<td>( \Gamma, A \vdash \Delta ) \rightarrow \rightarrow \Gamma \vdash \neg A, \Delta ) \rightarrow \rightarrow \Gamma \vdash A, \Delta ) \rightarrow \rightarrow \Gamma, \neg A \vdash \Delta )</td>
</tr>
</tbody>
</table>

### Logical group: binary dual connectives

#### Rules for conjunction

| \( \Gamma \vdash A, \Delta \) \rightarrow \rightarrow \Gamma' \vdash B, \Delta', \Delta' \rightarrow \rightarrow \Gamma, \Gamma' \vdash A \land B, \Delta, \Delta', \Delta', \Delta' \rightarrow \rightarrow \Gamma, \Delta, \Delta' \vdash A \land B \vdash \Delta, \Delta' \rightarrow \rightarrow \Gamma, A \land B \vdash \Delta, \Delta' |

#### Rules for disjunction

| \( \Gamma \vdash A, B, \Delta \) \rightarrow \rightarrow \Gamma \vdash A \lor B, \Delta \) \rightarrow \rightarrow \Gamma, \Delta \vdash A \lor B, \Delta \) \rightarrow \rightarrow \Gamma, \Gamma', A \lor B \vdash \Delta, \Delta', \Delta', \Delta' |

#### Multiplicative rules for conjunction

| \( \Gamma \vdash A, \Delta \) \rightarrow \rightarrow \Gamma' \vdash B, \Delta', \Delta' \rightarrow \rightarrow \Gamma, \Gamma' \vdash A \land B, \Delta, \Delta', \Delta', \Delta' \rightarrow \rightarrow \Gamma, \Delta, \Delta' \vdash A \land B \vdash \Delta, \Delta' \rightarrow \rightarrow \Gamma, A \land B \vdash \Delta, \Delta' |

#### Multiplicative rules for disjunction

| \( \Gamma \vdash A, B, \Delta \) \rightarrow \rightarrow \Gamma \vdash A \lor B, \Delta \) \rightarrow \rightarrow \Gamma, \Delta \vdash A \lor B, \Delta \) \rightarrow \rightarrow \Gamma, \Gamma', A \lor B \vdash \Delta, \Delta', \Delta', \Delta' |

#### Additive rules for conjunction

| \( \Gamma \vdash A, \Delta \) \rightarrow \rightarrow \Gamma' \vdash B, \Delta', \Delta' \rightarrow \rightarrow \Gamma, \Gamma' \vdash A \land B, \Delta, \Delta', \Delta', \Delta' \rightarrow \rightarrow \Gamma, \Delta, \Delta' \vdash A \land B \vdash \Delta, \Delta' \rightarrow \rightarrow \Gamma, A \land B \vdash \Delta, \Delta' |

#### Additive rules for disjunction

| \( \Gamma \vdash A, B, \Delta \) \rightarrow \rightarrow \Gamma \vdash A \lor B, \Delta \) \rightarrow \rightarrow \Gamma, \Delta \vdash A \lor B, \Delta \) \rightarrow \rightarrow \Gamma, \Gamma', A \lor B \vdash \Delta, \Delta', \Delta', \Delta' |

---

J-B Joinet (IRPhiL, Lyon 3)  
Introductory school in Linear Logic  
INTRO |  
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**Logical group : unary connectives (negation)**

<table>
<thead>
<tr>
<th>Negation</th>
<th></th>
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<tbody>
<tr>
<td>[\Gamma, A \vdash \Delta]</td>
<td>[\Gamma \vdash \neg A, \Delta]</td>
</tr>
<tr>
<td>[\neg \Gamma \vdash \neg A, \Delta]</td>
<td>[\neg \Gamma, \neg A \vdash \Delta]</td>
</tr>
</tbody>
</table>

---

**Logical group : binary dual connectives**

**rules for conjunction**

<table>
<thead>
<tr>
<th>[\Gamma \vdash A, \Delta]</th>
<th>[\Gamma' \vdash B, \Delta']</th>
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</thead>
<tbody>
<tr>
<td>[\Gamma, \Gamma' \vdash A \land B, \Delta, \Delta']</td>
<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>[\Gamma, A, B \vdash \Delta]</th>
</tr>
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<tbody>
<tr>
<td>[\Gamma, A \land B \vdash \Delta]</td>
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**rules for disjunction**

<table>
<thead>
<tr>
<th>[\Gamma \vdash A, B, \Delta]</th>
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<tbody>
<tr>
<td>[\Gamma \vdash A \lor B, \Delta]</td>
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<table>
<thead>
<tr>
<th>[\Gamma, A \vdash \Delta]</th>
<th>[\Gamma', B \vdash \Delta']</th>
</tr>
</thead>
<tbody>
<tr>
<td>[\Gamma, \Gamma', A \lor B \vdash \Delta, \Delta']</td>
<td></td>
</tr>
</tbody>
</table>
### Logical group: unary connectives (negation)

**Negation**

\[
\begin{align*}
\Gamma, A & \vdash \Delta \\
\Gamma & \vdash \neg A, \Delta \\
\end{align*}
\]

\[
\begin{align*}
\neg & \vdash \Gamma, \neg A \vdash \Delta
\end{align*}
\]

### Logical group: binary dual connectives

**Rules for conjunction**

\[
\begin{align*}
\Gamma & \vdash A, \Delta \\
\Gamma' & \vdash B, \Delta' \\
\Gamma, \Gamma' & \vdash A \land B, \Delta, \Delta'
\end{align*}
\]

\[
\Gamma, A, B \vdash \Delta
\]

**Rules for disjunction**

\[
\begin{align*}
\Gamma & \vdash A, B, \Delta \\
\Gamma & \vdash A \lor B, \Delta
\end{align*}
\]

\[
\begin{align*}
\Gamma, A & \vdash \Delta \\
\Gamma', B & \vdash \Delta' \\
\Gamma, \Gamma', A \lor B & \vdash \Delta, \Delta'
\end{align*}
\]

**Rules for conjunction**

\[
\begin{align*}
\Gamma & \vdash A, \Delta \\
\Gamma & \vdash B, \Delta \\
\Gamma & \vdash A \land B, \Delta
\end{align*}
\]

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\begin{align*}
\Gamma, A & \vdash \Delta \\
\Gamma, B & \vdash \Delta \\
\Gamma, A \land B & \vdash \Delta
\end{align*}
\]

**Rules for disjunction**

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\begin{align*}
\Gamma & \vdash A, \Delta \\
\Gamma & \vdash B, \Delta \\
\Gamma & \vdash A \lor B, \Delta
\end{align*}
\]

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\begin{align*}
\Gamma, A & \vdash \Delta \\
\Gamma, B & \vdash \Delta \\
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\end{align*}
\]

\[
\Gamma, A \lor B \vdash \Delta
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</tr>
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</tr>
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</tbody>
</table>

**Logical group: binary dual connectives**

**Multiplicative rules for conjunction**

<table>
<thead>
<tr>
<th>$\Gamma \vdash A, \Delta$</th>
<th>$\Gamma' \vdash B, \Delta'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma, \Gamma' \vdash A \land B, \Delta, \Delta'$</td>
<td></td>
</tr>
<tr>
<td>$\Gamma \land B \vdash \Delta$</td>
<td></td>
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</table>

**Multiplicative rules for disjunction**

<table>
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<tr>
<th>$\Gamma \vdash A, B, \Delta$</th>
</tr>
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<tbody>
<tr>
<td>$\Gamma \vdash A \lor B, \Delta$</td>
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</table>

**Additive rules for conjunction**

<table>
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<tr>
<th>$\Gamma \vdash A, \Delta$</th>
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<tbody>
<tr>
<td>$\Gamma \vdash A \land B, \Delta$</td>
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</tr>
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<td></td>
</tr>
<tr>
<td>$\Gamma, A \lor B \vdash \Delta$</td>
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</tbody>
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### Logical group: unary connectives (negation)

**Negation**

<table>
<thead>
<tr>
<th>$\Gamma, A \vdash \Delta$</th>
<th>$\Gamma \vdash \neg A, \Delta$</th>
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<tbody>
<tr>
<td>$\neg \Gamma \vdash \neg A, \Delta$</td>
<td>$\neg \Gamma, \neg A \vdash \Delta$</td>
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### Logical group: binary dual connectives

#### Rules for conjunction

<table>
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#### Rules for disjunction

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<th>$\Gamma \vdash A, B, \Delta$</th>
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<td>$\Gamma \vdash A \lor B, \Delta$</td>
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### Logical group: unary connectives (negation)

**Negation**

\[
\begin{align*}
\Gamma, A &\vdash \Delta \\
\Gamma &\vdash \neg A, \Delta
\end{align*}
\]

\[
\begin{align*}
\Gamma &\vdash A, \Delta \\
\neg \Gamma, \neg A &\vdash \Delta
\end{align*}
\]

### Logical group: binary dual connectives

#### Multiplicative rules for conjunction

\[
\begin{align*}
\Gamma &\vdash A, \Delta \\
\Gamma' &\vdash B, \Delta' \\
\Gamma, \Gamma' &\vdash A \land B, \Delta, \Delta'
\end{align*}
\]

\[
\begin{align*}
\Gamma, \Gamma' &\vdash A \land B, \Delta, \Delta'
\end{align*}
\]

#### Multiplicative rules for disjunction

\[
\begin{align*}
\Gamma &\vdash A, B, \Delta \\
\Gamma &\vdash A \lor B, \Delta
\end{align*}
\]

\[
\begin{align*}
\Gamma, \Gamma' &\vdash A \lor B, \Delta, \Delta'
\end{align*}
\]

#### Additive rules for conjunction

\[
\begin{align*}
\Gamma &\vdash A, \Delta \\
\Gamma &\vdash B, \Delta \\
\Gamma &\vdash A \land B, \Delta
\end{align*}
\]

\[
\begin{align*}
\Gamma, \Gamma' &\vdash A \land B, \Delta, \Delta'
\end{align*}
\]

#### Additive rules for disjunction

\[
\begin{align*}
\Gamma &\vdash A, B, \Delta \\
\Gamma &\vdash A \lor B, \Delta \\
\Gamma &\vdash A \lor B, \Delta
\end{align*}
\]

\[
\begin{align*}
\Gamma, \Gamma' &\vdash A \lor B, \Delta, \Delta'
\end{align*}
\]

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\begin{align*}
\Gamma &\vdash A \lor B, \Delta
\end{align*}
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\[
\begin{align*}
\Gamma, \Gamma' &\vdash A \lor B, \Delta, \Delta'
\end{align*}
\]
### Logical group: unary connectives (negation)

**Negation**

\[
\Gamma, A \vdash \Delta \\
\vdash \Gamma, \neg A, \Delta \\
\neg \Gamma \vdash \Delta
\]

### Logical group: binary dual connectives

**Multiplicative**

**Rules for conjunction**

\[
\Gamma \vdash A, \Delta \\
\Gamma' \vdash B, \Delta' \\
\Gamma, \Gamma' \vdash A \land B, \Delta, \Delta'
\]

**Rules for disjunction**

\[
\Gamma \vdash A, B, \Delta \\
\vdash \Gamma, \Gamma' \vdash A \lor B, \Delta
\]

**Additive**

**Rules for conjunction**

\[
\Gamma \vdash A, \Delta \\
\Gamma \vdash B, \Delta \\
\vdash \Gamma \vdash A \land B, \Delta
\]

**Rules for disjunction**

\[
\Gamma \vdash A, \Delta \\
\Gamma \vdash B, \Delta \\
\vdash \Gamma \vdash A \lor B, \Delta
\]
**Logical group : unary connectives (negation)**

**Negation**

\[ \Gamma, A \vdash \Delta \]

\[ \Gamma \vdash \neg A, \Delta \]

\[ \neg \Gamma \vdash \neg A, \Delta \]

\[ \Gamma, \neg A \vdash \Delta \]

**Logical group : binary dual connectives**

### Multiplicative rules for conjunction

\[ \Gamma \vdash A, \Delta \]

\[ \Gamma' \vdash B, \Delta' \]

\[ \Gamma, \Gamma' \vdash A \land B, \Delta, \Delta' \]

\[ \Gamma \land \Gamma' \vdash A \land B, \Delta, \Delta' \]

### Multiplicative rules for disjunction

\[ \Gamma \vdash A, B, \Delta \]

\[ \Gamma \vdash A \lor B, \Delta \]

\[ \Gamma, \Gamma' \vdash A \lor B \]

\[ \Gamma, \Gamma' \vdash A \lor B, \Delta, \Delta' \]

### Additive rules for conjunction

\[ \Gamma \vdash A, \Delta \]

\[ \Gamma \vdash B, \Delta \]

\[ \Gamma \vdash A \land B, \Delta \]

\[ \Gamma \land \Delta \vdash A \land B \]

\[ \Gamma \land \Delta \vdash A \land B \]

\[ \Gamma \land \Delta \vdash A \land B \]

### Additive rules for disjunction

\[ \Gamma \vdash A, \Delta \]

\[ \Gamma \vdash B, \Delta \]

\[ \Gamma \vdash A \lor B, \Delta \]

\[ \Gamma \lor \Delta \vdash A \lor B \]

\[ \Gamma \lor \Delta \vdash A \lor B \]

\[ \Gamma \lor \Delta \vdash A \lor B \]
### Logical group: unary connectives (negation)

#### Negation

<table>
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<tr>
<th>Rule</th>
<th>Description</th>
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<tbody>
<tr>
<td>[ \Gamma, A \vdash \Delta ]</td>
<td>[ \Gamma \vdash \neg A, \Delta ]</td>
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### Logical group: binary dual connectives

#### Multiplicative rules for conjunction

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<tbody>
<tr>
<td>[ \Gamma \vdash A, \Delta ]</td>
<td>[ \Gamma', B \vdash \Delta', \gamma' ]</td>
</tr>
<tr>
<td>[ \Gamma \vdash A, B, \Delta ]</td>
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#### Multiplicative rules for disjunction

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<tr>
<td>[ \Gamma \vdash A, B, \Delta ]</td>
<td>[ \Gamma \vdash A \lor B, \Delta ]</td>
</tr>
<tr>
<td>[ \Gamma, A \vdash \Delta ]</td>
<td>[ \Gamma', B \vdash \Delta' ]</td>
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#### Additive rules for conjunction

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<td>[ \Gamma \vdash A \lor B, \Delta ]</td>
</tr>
</tbody>
</table>
## Logical group: unary connectives (negation)

Negation

\[
\Gamma, A \vdash \Delta \\
\Gamma \vdash \neg A, \Delta \quad \neg \Gamma \vdash A, \Delta
\]

## Logical group: binary dual connectives

### Multiplicative

**rules for conjunction**

\[
\Gamma \vdash A, \Delta \\
\Gamma' \vdash B, \Delta' \\
\Gamma, \Gamma' \vdash A \land B, \Delta, \Delta' \\
\]

**rules for disjunction**

\[
\Gamma \vdash A, B, \Delta \\
\Gamma \vdash A \lor B, \Delta \\
\]

### Additive

**rules for conjunction**

\[
\Gamma \vdash A, \Delta \\
\Gamma \vdash B, \Delta \\
\Gamma \vdash A \land B, \Delta \\
\Gamma \vdash A \lor B, \Delta \\
\]

**rules for disjunction**

\[
\Gamma \vdash A, \Delta \\
\Gamma \vdash B, \Delta \\
\Gamma \vdash A \lor B, \Delta \\
\]

---

**Introductory school in Linear Logic**

INTRODUCTION

LL2016
PART 3

Classical logic decomposed
Additive and Multiplicative styles equivalent modulo the structural rules

- In presence of the structural rules, the distinction between multiplicative and additive styles degenerates, i.e. one has:

\[
\begin{align*}
A \lor B &\equiv A \lor B \\
A \land B &\equiv A \land B
\end{align*}
\]

Indeed (for instance):

\[
\begin{align*}
A \vdash A, B \vdash B \\
A \vdash A, B \vdash B, A \lor B \vdash A \lor B, B \lor A \vdash A \lor B \\
A \vdash A, B \vdash B, A \lor B \vdash A \lor B, B \lor A \vdash A \lor B
\end{align*}
\]
Additive and Multiplicative styles equivalent modulo the structural rules

In presence of the structural rules, the distinction between multiplicative and additive styles degenerates, i.e. one has:

\[
A \lor B \equiv A \lor B
\]
\[
A \land B \equiv A \land B
\]

Indeed (for instance):

\[
\frac{A \vdash A}{A, B \vdash A} \quad \frac{B \vdash B}{A, B \vdash B}
\]
\[
A \lor B \vdash A, B
\]
\[
A \lor B \vdash A \lor B
\]
\[
\frac{A \vdash A}{A \lor B, A \vdash B}
\]
\[
A \lor B \vdash A \lor B
\]
\[
A \lor B \vdash A \lor B
\]
\[
\frac{A \lor B \vdash A \lor B, A \lor B}{A \lor B \vdash A \lor B}
\]
\[
\frac{m}{A \vdash A}
\]
\[
\frac{m}{B \vdash B}
\]
\[
\frac{m}{A \lor B \vdash A \lor B, A \lor B}
\]
\[
\frac{m}{A \lor B \vdash A \lor B, A \lor B}
\]
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\]
\[
\frac{m}{A \lor B \vdash A \lor B, A \lor B}
\]
\[
\frac{m}{A \lor B \vdash A \lor B, A \lor B}
\]
Additive and Multiplicative styles equivalent modulo the structural rules

- But structural rules are needed for that (easy to show once eliminability of cut is proved).
Additive and Multiplicative styles equivalent modulo the structural rules

- But structural rules are needed for that (easy to show once eliminability of cut is proved).

- So in the fragment of LK with no structural rules, the rules with the various style define genuine (non equivalent) connectives.
Additive and Multiplicative styles equivalent modulo the structural rules

- But structural rules are needed for that (easy to show once eliminability of cut is proved).

- So in the fragment of LK with no structural rules, the rules with the various style define genuine (non equivalent) connectives.

- Notation and terminology:
  - $\land^m$ noted $\otimes$ (“tensor”, “times”)
  - $\lor^m$ noted $\otimes$ (“par”, “co-tensor”)
  - $\land^a$ noted $&$ (“with”)
  - $\lor^a$ noted $\oplus$ (“plus”)
MALL: Identity group

Identity group

Identity axiom

\[ \text{ax} \quad A \vdash A \]

Cut rule

\[ \frac{\Gamma_1 \vdash \Delta_1, A \quad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \quad \text{cut} \]
### Logical group: unary connectives (negation)

**Negation**

<table>
<thead>
<tr>
<th>$\Gamma, A \vdash \Delta$</th>
<th>$\Gamma \vdash \neg A, \Delta$</th>
</tr>
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<tbody>
<tr>
<td>$\neg \Gamma \vdash \neg A, \Delta$</td>
<td>$\Gamma, \neg A \vdash \Delta$</td>
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</table>

### Logical group: binary dual connectives

**Multiplicative conjunction**

$\otimes$, "Tensor"

<table>
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<tr>
<th>$\Gamma \vdash A, \Delta$</th>
<th>$\Gamma' \vdash B, \Delta'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'$</td>
<td></td>
</tr>
</tbody>
</table>

**Multiplicative disjunction**

$\wedge$, "Par"

<table>
<thead>
<tr>
<th>$\Gamma \vdash A, B, \Delta$</th>
<th>$\Gamma' \vdash \Delta'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma, \Gamma', A \wedge B \vdash \Delta, \Delta'$</td>
<td></td>
</tr>
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</table>

**Additive conjunction**

$\&$, "With"

<table>
<thead>
<tr>
<th>$\Gamma \vdash A, \Delta$</th>
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<tbody>
<tr>
<td>$\Gamma \vdash A &amp; B, \Delta$</td>
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**Additive disjunction**

$\oplus$, "Plus"

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<tr>
<th>$\Gamma \vdash A, \Delta$</th>
<th>$\Gamma \vdash B, \Delta$</th>
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</thead>
<tbody>
<tr>
<td>$\Gamma \vdash A \oplus B, \Delta$</td>
<td>$\Gamma \vdash A \oplus B, \Delta$</td>
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</table>
### Logical group: unary connectives (negation)

**Negation**

\[
\begin{align*}
\Gamma, A \vdash \Delta & \quad \rightarrow \\
\Gamma \vdash \neg A, \Delta & \\
\Gamma \vdash A, \Delta & \quad \leftarrow
\end{align*}
\]

### Logical group: binary dual connectives

**Multiplicative conjunction**

\(\otimes\), “Tensor”

\[
\begin{align*}
\Gamma \vdash A, \Delta & \quad \Gamma' \vdash B, \Delta' \\
\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta' & \\
\end{align*}
\]

**Multiplicative disjunction**

\(\uplus\), “Par”

\[
\begin{align*}
\Gamma \vdash A, B, \Delta & \\
\Gamma \vdash A \uplus B, \Delta & \\
\end{align*}
\]

**Additive conjunction**

\(&\), “With”

\[
\begin{align*}
\Gamma \vdash A, \Delta & \quad \Gamma \vdash B, \Delta \\
\Gamma \vdash A \& B, \Delta & \\
\end{align*}
\]

**Additive disjunction**

\(\oplus\), “Plus”

\[
\begin{align*}
\Gamma \vdash A, \Delta & \quad \Gamma \vdash B, \Delta \\
\Gamma \vdash A \oplus B, \Delta & \\
\Gamma \vdash A \oplus B, \Delta & \\
\Gamma, A \oplus B \vdash \Delta
\end{align*}
\]
### Logical group: Unary Connectives (Negation)

#### Negation

\[
\Gamma, A \vdash \Delta \\
\Gamma \vdash \neg A, \Delta
\]

\[
\neg \Gamma \vdash A, \Delta \\
\Gamma, \neg A \vdash \Delta
\]

### Logical group: Binary Dual Connectives

#### Multiplicative Conjunction

\[
\Gamma \vdash A, \Delta \\
\Gamma' \vdash B, \Delta' \\
\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'
\]

#### Multiplicative Disjunction

\[
\Gamma, A, B \vdash \Delta \\
\Gamma, A \otimes B \vdash \Delta
\]

#### Additive Conjunction

\[
\Gamma \vdash A, \Delta \\
\Gamma \vdash B, \Delta \\
\Gamma \vdash A \& B, \Delta
\]

#### Additive Disjunction

\[
\Gamma \vdash A, \Delta \\
\Gamma \vdash B, \Delta \\
\Gamma \vdash A \oplus B, \Delta
\]

#### J-B Joinet (IRPhiL, Lyon 3)

- Introductory school in Linear Logic
The language is enriched with four 0-ary connectives (thus formulas):

- **Multiplicative ones**: $1$ and $\bot$
- **Additive ones**: $0$ and $\top$

They are neutrals:

- $1$ is provably neutral for $\otimes$
- $\bot$ is provably neutral for $\odot$
- $\bot$ is provably neutral for $\&$
- $0$ is provably neutral for $\oplus$
- $\top$ is provably neutral for $\land$
- $\top$ is provably neutral for $\lor$

No left intro for $\top$
No right rule for $0$
The language is enriched with four 0-ary connectives (thus formulas):

Multiplicative ones: $1$ and $\bot$

Additive ones: $0$ and $\top$

### MALL Logical group continued: Neutrals

<table>
<thead>
<tr>
<th>0-ary multiplicatives (neutrals)</th>
<th>0-ary additives (neutrals)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$\top$</td>
</tr>
<tr>
<td>$\Gamma \vdash \Delta$</td>
<td>$\top$</td>
</tr>
<tr>
<td>$\Gamma, 1 \vdash \Delta$</td>
<td>No left intro for $\top$</td>
</tr>
<tr>
<td>$\Gamma, \bot \vdash \Delta$</td>
<td>No right rule for $0$</td>
</tr>
<tr>
<td>$\bot$</td>
<td>$\bot$</td>
</tr>
</tbody>
</table>

No right rule for $0$

No left intro for $\top$
MALL : adding 0-ary connectives (neutrals)

The language is enriched with four 0-ary connectives (thus formulas):

- **Multiplicative ones**: 1 and \( \perp \)
- **Additive ones**: 0 and \( \top \)

### MALL Logical group continued: Neutrals

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</tr>
<tr>
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<td>( \top )</td>
</tr>
<tr>
<td>( \Gamma, \top \vdash \Delta )</td>
<td>( \top )</td>
</tr>
<tr>
<td>( \Gamma, \perp \vdash \Delta )</td>
<td>( \top )</td>
</tr>
</tbody>
</table>

They are neutrals:

- 1 is provably neutral for \( \otimes \)
- \( \perp \) is provably neutral for \( \oslash \)
- \( \top \) is provably neutral for \( \& \)
- 0 is provably neutral for \( \oplus \)
MLL, ALL, MALL are computational fragments

Each of MLL, ALL (and thus MALL) satisfies:

- **Cut-elimination** (no contractions $\Rightarrow$ low complexity process)
MLL, ALL, MALL are computational fragments

Each of MLL, ALL (and thus MALL) satisfies:

- **Cut-elimination** (no contractions ⇒ low complexity process)

- **Atomization of axioms**, i.e. Identities are canonically provable from atomic initial sequents:

\[
\frac{A \vdash A}{A, B \vdash A \otimes B} \otimes
\]
\[
\frac{A \otimes B \vdash A \otimes B}{A \otimes B \vdash A \otimes B}
\]

\[
\frac{A \vdash A}{A & B \vdash A}
\frac{B \vdash B}{A & B \vdash B}
\frac{A \& B \vdash A}{A & B \vdash A & B}
\frac{A \& B \vdash B}{A & B \vdash A & B}
\]

\[
\frac{A \vdash A}{A \otimes B \vdash A, B}
\frac{B \vdash B}{A \otimes B \vdash A, B}
\frac{A \otimes B \vdash A \otimes B}{A \otimes B \vdash A \otimes B}
\]

\[
\frac{A \vdash A}{A \oplus B \vdash A}
\frac{B \vdash B}{A \oplus B \vdash B}
\frac{A \oplus B \vdash A \oplus B}{A \oplus B \vdash A \oplus B}
\]

\[
\frac{1 \vdash 1}{1 \vdash 1}
\frac{1 \vdash 1}{1 \vdash 1}
\]

\[
\frac{\top \vdash \top}{\top \vdash \top}
\frac{\bot \vdash \bot}{\bot \vdash \bot}
\frac{0 \vdash 0}{0 \vdash 0}
\]
De Morgan dualities

Each fragment MLL, ALL (and thus MALL) satisfies De Morgan equivalences:

\(- (A \otimes B) \equiv_{\text{MLL}} \neg A \bowtie \neg B\)
\(- (A \bowtie B) \equiv_{\text{MLL}} \neg A \otimes \neg B\)

\(- 1 \equiv_{\text{MLL}} \bot\)
\(- \bot \equiv_{\text{MLL}} 1\)

\(- (A \& B) \equiv_{\text{ALL}} \neg A \oplus \neg B\)
\(- (A \oplus B) \equiv_{\text{ALL}} \neg A \& \neg B\)

\(- \top \equiv_{\text{ALL}} 0\)
\(- 0 \equiv_{\text{ALL}} \top\)
De Morgan dualities

Each fragment MLL, ALL (and thus MALL) satisfies De Morgan equivalences:

\[
\neg(A \otimes B) \equiv_{\text{MLL}} \neg A \nabla \neg B
\]
\[
\neg(A \nabla B) \equiv_{\text{MLL}} \neg A \otimes \neg B
\]
\[
\neg 1 \equiv_{\text{MLL}} \bot
\]
\[
\neg \bot \equiv_{\text{MLL}} 1
\]
\[
\neg (A \& B) \equiv_{\text{ALL}} \neg A \oplus \neg B
\]
\[
\neg (A \oplus B) \equiv_{\text{ALL}} \neg A \& \neg B
\]
\[
\neg \top \equiv_{\text{ALL}} 0
\]
\[
\neg 0 \equiv_{\text{ALL}} \bot
\]

Pairs of mutually dual connectives:

\[\otimes / \nabla \quad \oplus / \& \quad 1 / \bot \quad 0 / \top \quad (\text{and } \forall / \exists)\]
De Morgan dualities

- Each fragment MLL, ALL (and thus MALL) satisfies De Morgan equivalences:
  \[ \neg(A \otimes B) \equiv_{\text{MLL}} \neg A \otimes \neg B \]
  \[ \neg 1 \equiv_{\text{MLL}} \bot \quad \neg \bot \equiv_{\text{MLL}} 1 \]
  \[ \neg(A \& B) \equiv_{\text{ALL}} \neg A \oplus \neg B \]
  \[ \neg(T \equiv_{\text{ALL}} 0 \quad \neg 0 \equiv_{\text{ALL}} T \]

- Pairs of mutually dual connectives:
  \[ \otimes / \otimes \quad \oplus / \& \quad 1 / \bot \quad 0 / \top \quad (\text{and } \forall / \exists) \]

- Symmetry or chattering? Up to the exchanges left/right and the exchange of dual connectives, everything is said twice. For instance:

\[
\frac{A \vdash A \quad B \vdash B}{A \otimes B \vdash A \otimes B} \quad \otimes \quad \frac{A \vdash A \quad B \vdash B}{A \& B \vdash A \& B} \quad \&
\]

\[
\frac{A \vdash A \quad B \vdash B}{A \otimes B \vdash A \otimes B} \quad \otimes \quad \frac{A \vdash A \quad B \vdash B}{A \& B \vdash A \& B} \quad \&
\]

\[
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\]

\[
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\]

\[
\frac{A \vdash A \quad B \vdash B}{A \otimes B \vdash A \otimes B} \quad \otimes \quad \frac{A \vdash A \quad B \vdash B}{A \& B \vdash A \& B} \quad \&
\]
Ceasing chattering : toward monolateral sequent calculus

- **Goal**: to divide the number of rules for binary connectives by two.

- **Tool**: replace negation as a unary function by the binary relation of duality.

Step 1: we change the notion of formulas (our old set of formulas will be quotiented by de Morgan equivalences):

- Negation no more present as a connective, but as a defined operation \( \bot \).
- Atoms come by pairs: each atom \( \text{X} \) comes with its dual noted \( \text{X} \bot \).
- \( (\text{X} \bot) \bot = \text{X} \).
- The dual \( \text{A} \bot \) of \( \text{A} \) is "the De Morganized" form of \( \neg \text{A} \).

For instance: if \( \text{A} = (\text{X} \otimes (\text{Y} \bot \& \text{Z})) \), then \( \text{A} \bot \) denotes \( (\text{X} \bot ` (\text{Y} \oplus \text{Z} \bot)) \).

Step 2: we fold the left side on the right side up; and, in the "Identity group", we replace identity constraints on formulas (the one on the hypothesis side, the one on the conclusions side) by duality constraints (on the conclusions side).
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Step 1: we change the notion of formulas (our old set of formulas will be quotiented by de Morgan equivalences):

- Negation no more present as a connective, but as a defined operation \((.)^\perp\)
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- \((X^\perp)^\perp = X\)
- The dual \(A^\perp\) of \(A\) is “the De Morganized” form of \(\neg A\)
- For instance: if \(A = (X \otimes (Y^\perp \& Z))\), then \(A^\perp\) denotes \((X^\perp \bowtie (Y \oplus Z^\perp))\)
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MALL, MLL, ALL as monolateral systems

Identity group (better called: duality group)

\[ \frac{\vdash A, A^\perp}{\vdash \Gamma, A, \Delta, A^\perp} \text{ ax} \]

\[ \vdash \Gamma, A \quad \vdash \Delta, A^\perp \]

\[ \frac{\vdash \Gamma, \Delta}{\vdash \Gamma, \Delta} \text{ cut} \]

Logical group (dual connectives)

Multiplicatives

Binary multiplicatives

\[ \vdash A, \Gamma \quad \vdash B, \Delta \]

\[ \vdash A \otimes B, \Gamma, \Delta \] \quad \text{⊗}

\[ \vdash A \triangledown B, \Gamma \] \quad \text{γ}

0-ary multiplicatives (neutrals)

\[ \vdash 1 \]

Additives

\[ \vdash A, \Gamma \quad \vdash B, \Gamma \]

\[ \vdash A \& B, \Gamma \] \quad \text{&}

\[ \vdash A \oplus B, \Gamma \] \quad \text{⊕}

\[ \vdash B, \Gamma \]

0-ary additives (neutrals)

\[ \vdash \top, \Gamma \]

No rule for \( 0 \)
Multiplicatives and Additive now coexist while being separated.

A price is paid: forgetting the structural rules = losing algorithmic expressivity.

Indeed, dynamically (i.e., w.r.t. the cut-elimination process):
- Contraction rule = duplication (the source of the complexity of cut-elimination)
- Weakening rule = erasing

In order to recover algorithmic richness and complexity: to reintroduce some structural rules is required.

Problem: how to reintroduce them avoiding the collapse of (multiplicative/additive) styles?

LL answer:
- Introduce "aspect" in the logical language
- Aspect: a category coming from natural languages grammar
  - In Indo-European languages generally implemented by tenses: perfect vs imperfect
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Imperfect = durable (reusable). Perfect = Instantaneous (non reusable)
Exponentials

- Imperfect = durable (reusable). Perfect = Instantaneous (non reusable)
- Reminder: the LK contraction rule:

\[
\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \text{ ctr}
\]
Exponentials

- Imperfect = durable (reusable). Perfect = Instantaneous (non reusable)
- Reminder: the LK contraction rule:
  \[
  \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \quad \text{ctr}
  \]
- From a proof search point of view (i.e. a Bottom-up reading), that rule says: one can “save” \(A\). In other words: any \(A\) is reusable.
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- Structural rules of LL:

\[
\frac{\Gamma, ?A, ?A}{\Gamma, ?A} \quad \text{ctr}
\]

\[
\frac{\Gamma}{\Gamma, ?A} \quad \text{w}
\]
Exponentials

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\]

\[
\frac{\vdash \Gamma}{\vdash \Gamma, ?A} \quad \text{w}
\]

- The dual of ? will be noted ! (they are unary connectives or “modalities”)
Identity group (duality group)

\[ \vdash A, A^\perp \quad \text{ax} \]

\[ \vdash \Gamma, A \quad \vdash \Delta, A^\perp \quad \text{cut} \]

Logical group

Multiplicatives

2-ary multiplicatives

\[ \vdash A, \Gamma \quad \vdash B, \Delta \]

\[ \vdash A \otimes B, \Gamma, \Delta \quad \otimes \]

0-ary multiplicatives (neutrals)

\[ \vdash 1 \]

Additives

2-ary additives

\[ \vdash A, \Gamma \quad \vdash B, \Gamma \]

\[ \vdash A \& B, \Gamma \quad \& \]

0-ary additives (neutrals)

\[ \vdash \Gamma, \top \]

Exponentials

No rule for 0

\[ \vdash ?\Gamma, A \]

\[ \vdash ?\Gamma, !A \]

Structural group

\[ \vdash \Gamma, ?A, ?A \quad \text{ctr} \]

\[ \vdash \Gamma, ?A \quad \text{w} \]
Comments on the introduction rules for exponentials

» ! and ? are dual connectives
Comments on the introduction rules for exponentials

- ! and ? are dual connectives
- The introduction rule for respectively ! and ? are the same as the rules for □ and ◊ in the sequent calculus for modal logic S4. But the structural rules specific to LL makes it differ from S4.
Comments on the introduction rules for exponentials

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- The rule (called “promotion”) for introducing ! is contextual (global): one has to think to it as pointing out the corresponding subproof (premise of the rule).
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- a good way to figure this (and this anticipates the proofnets with exponentials) is to picture the corresponding subproof in a box.
Comments on the introduction rules for exponentials

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- The rule (called “promotion”) for introducing ! is contextual (global): one has to think to it as pointing out the corresponding subproof (premise of the rule).
- A good way to figure this (and this anticipates the proofnets with exponentials) is to picture the corresponding subproof in a box.
- When one does so, the formula !A is called the main door of the box, the formulas in ?Γ are called the auxiliary doors of that box.
The box representation is particularly efficacious to explain the dynamical rôle of exponentials.
The dynamic of exponentials

- The box representation is particularly efficacious to explain the dynamical rôle of exponentials.

- During the cut-elimination process:
  - when the main door \(!A\) of a box is confronted via a cut to a contracted formula \(?A^\perp\), the box is duplicated (note that the auxiliary doors being prefixed with a \(?\), they can be contracted to preserve the conclusion of the proof).
The dynamic of exponentials

- The box representation is particularly efficacious to explain the dynamical rôle of exponentials
- During the cut-elimination process:
  - when the main door $!A$ of a box is confronted via a cut to a contracted formula $?A^\bot$, the box is duplicated (note that the auxiliary doors being prefixed with a $?$, they can be contracted to preserve the conclusion of the proof)
  - when the main door $!A$ of a box is confronted via a cut to a weakened formula $?A^\bot$, the box is erased, including its content (note again that the auxiliary doors being prefixed with a $?$, they can be reintroduced by weakenings to preserve the conclusion of the proof)
The dynamic of exponentials

- The box representation is particularly efficacious to explain the dynamical rôle of exponentials

- During the cut-elimination process:
  - when the main door !A of a box is confronted via a cut to a contracted formula ?A⊥, the box is duplicated (note that the auxiliary doors being prefixed with a ?, they can be contracted to preserve the conclusion of the proof)
  - when the main door !A of a box is confronted via a cut to a weakened formula ?A⊥, the box is erased, including its content (note again that the auxiliary doors being prefixed with a ?, they can be reintroduced by weakenings to preserve the conclusion of the proof)

- Knowing that contracted or weakened formulas are always prefixed by a ?, only boxes can be so duplicated or erased. So, that the promotion rule is but a “declaration” that the concerned subproof will be potentially subject to “non linear” manipulations (duplications, erasures) during the process.
Boxes induce a natural stratification of proofs, so that computations arise at a given depth (a depth which evolve during the process):
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- When the main door $!A$ of a box is confronted via a cut to a formula $?A^\perp$ being auxiliary door of another box, the first box will enter in the second one: the dynamic of exponential thus underline a subtile analyse of the modification of the depth at which substitutions happen.
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- When the main door \( !A \) of a box is confronted via a cut to a derelicted formula \(?A^\perp\), the “declaration” evoked in the previous item is forgotten (dynamically, the dereliction rule is an instruction producing the erasure of the box declaration - but not of the content of the box).
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Notice that variants of linear logic take opportunity of these decomposition to freeze those possibilities, hence designing sub-logics of LL in which the computation are “tamed” (implicit complexity).
Conclusion

▶ Controlling the non linear operations in computation:
  ▶ made the additives and the multiplicatives distinction emerge
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  - made the additives and the multiplicatives distinction emerge
  - gave, through exponentials, an analyse of the substitution process (focusing on specific non linear manipulations and depth modification mechanisms)
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- gave, through exponentials, an analyse of the substitution process (focusing on specific non linear manipulations and depth modification mechanisms)

Initial slogan explained:

Linear Logic is Classical logic, but decomposed and observed through the microscope of the computational point of view on proofs.