Introductory school in Linear Logic

INTRODUCTION

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LL2016, Lyon
school: 7 and 8 November 2016

2016: the words 'Linear Logic' cover a huge field

- Various chapters and tools, among which:
  - Sequent calculi
  - Proofs-nets
  - Denotational semantics
  - Game semantics and Ludics
  - Geometry of interaction
  - Implicit computational complexity
  - Categorical approaches
  - etc
- Various proofs systems
  - fragments, variants, extensions of the original LL : e.g. MLL, ALL, MALL, MELL, LL, ELL, LLL etc
  - Linear Logics

1986: birth of LL

- Due to Jean-Yves Girard, a french logician
- LL borns at the meeting point of two scientific lines:
  - *Proof theory* (Logic)
  - *Computing theory* (Denotational semantics of programs)

What denoted the words "Linear Logic" at the very beginning?

- A sequent calculus logical system
- "Yet another formal logic" ? No.
- Linear Logic = Classical logic, but decomposed and observed through the microscope of the computational point of view on proofs
- Goal of this introduction: make understand what this means. . .

Make understand what it means that :

"Linear Logic is just **Classical logic** but **decomposed** through the **microscope** of **the computational point of view on proofs**"

PLAN :

- PART 1. Proofs in classical logic (static)
- PART 2. The computational point of view on proofs (dynamic)
- PART 3. Classical logic decomposed

PART 1
PROOFS IN CLASSICAL LOGIC

Important steps in the history of Proofs representation

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- Prove the consistency of formal methods (Peano axioms for arithmetics)
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- What is a proof ?
  - Hilbertian answer : a discourse respecting the rules of logic (local correctness)
  - Other possible answers : e.g. a universal strategy against any argumentative attack (dialectic answer)

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Let us start with Natural Deduction :

- Proofs as texts,
- deriving statements (formulas) from statements (formulas)

Propositional formulas

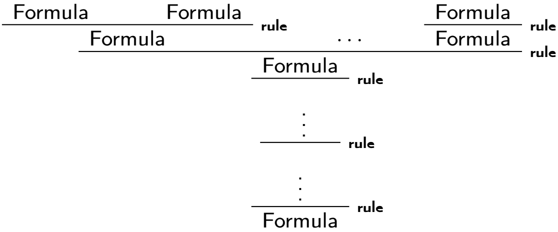
Example (a formula):

$$((\neg X \to Y) \wedge X) \vee \neg(Y \to \bot)$$

Formulas $A, B, C \ldots$ are inductively built :

- from elementary formulas :
  - atoms : X, Y, Z etc (atomic formulas)
  - and absurdum : $\bot$
- by applying :
  - the unary constructor negation : $\neg A$
  - the binary constructors :
    - conjunction: $A \wedge B$
    - disjunction: $A \vee B$
    - implication: $A \to B$

Natural Deduction: General shape of proofs (tree)



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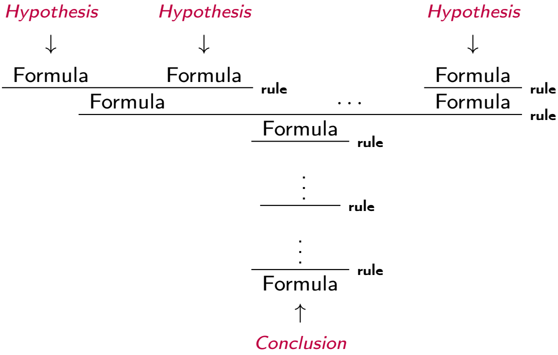

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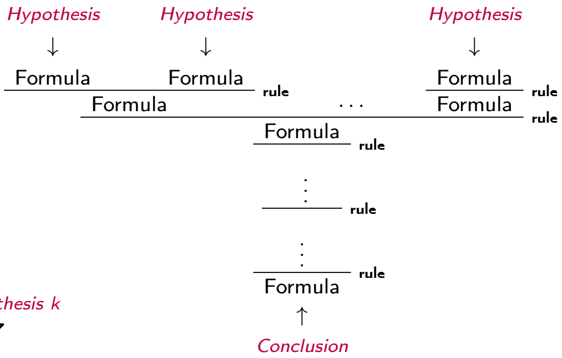
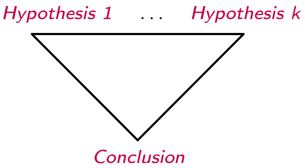
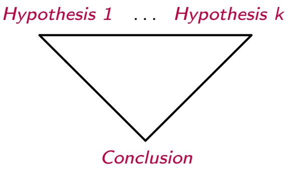




Proofs in Natural Deduction : an example

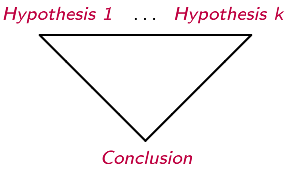
$$\dfrac{\dfrac{X \to (Y \to Z) \qquad X}{Y \to Z}\ {\to}\text{-élim} \qquad \dfrac{X \to Y \qquad X}{Y}\ {\to}\text{-élim}}{Z}\ {\to}\text{-élim}$$



Proofs in Natural Deduction : an example

Occurrence 1 ↓ (the first X) — Occurrence 2 ↓ (the second X)

$$\dfrac{\dfrac{X \rightarrow (Y \rightarrow Z) \qquad X}{Y \rightarrow Z}\ \rightarrow\text{-élim} \qquad \dfrac{X \rightarrow Y \qquad X}{Y}\ \rightarrow\text{-élim}}{Z}\ \rightarrow\text{-élim}$$



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Occurrence 1 ↓ Occurrence 2 ↓

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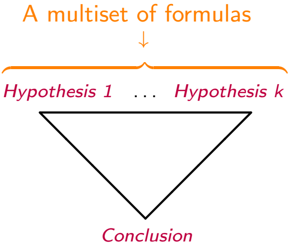

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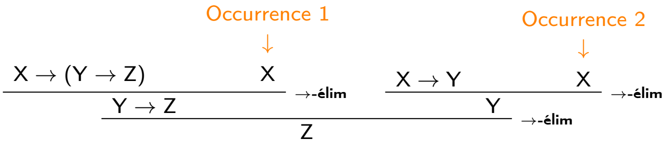
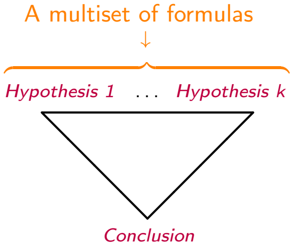


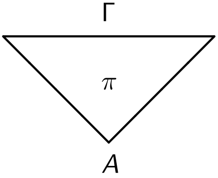


A **multiset** = a set "with repetitions" (no matter the order)

Representation of proofs in Natural Deduction

Notations:

- Formulas : A, B, C etc
- Multisets of formulas: Γ, Δ etc
- Proofs: π etc



Simply represented by

$$\begin{array}{c} \Gamma \\ \vdots\, \pi \\ A \end{array}$$

Rules

Introductions	Eliminations
$\dfrac{A \qquad\qquad B}{A \wedge B}\ \wedge\text{-intro}$	$\dfrac{A \wedge B}{A}\ \wedge\text{-elim (1)} \qquad \dfrac{A \wedge B}{B}\ \wedge\text{-elim (2)}$
	$\dfrac{A \to B \qquad\qquad A}{B}\ \to\text{-elim}$
$\dfrac{A}{A \vee B}\ \vee\text{-intro (1)} \qquad \dfrac{B}{A \vee B}\ \vee\text{-intro (2)}$	

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Remark. Rules are constructors for proofs (not transitions from formulas to formulas)

Rules

Introductions	Eliminations
$\dfrac{\begin{matrix}\Gamma \\ \vdots\, \pi_1 \\ A\end{matrix} \qquad \begin{matrix}\Gamma \\ \vdots\, \pi_2 \\ B\end{matrix}}{A \wedge B}\ \wedge\text{-intro}$	$\dfrac{\begin{matrix}\Gamma \\ \vdots\, \pi \\ A \wedge B\end{matrix}}{A}\ \wedge\text{-elim (1)} \qquad \dfrac{\begin{matrix}\Gamma \\ \vdots\, \pi \\ A \wedge B\end{matrix}}{B}\ \wedge\text{-elim (2)}$
	$\dfrac{\begin{matrix} \\ \vdots\, \pi_1 \\ A \rightarrow B\end{matrix} \qquad \begin{matrix}\Gamma \\ \vdots\, \pi_2 \\ A\end{matrix}}{B}\ \rightarrow\text{-elim}$
$\dfrac{\begin{matrix}\Gamma \\ \vdots\, \pi \\ A\end{matrix}}{A \vee B}\ \vee\text{-intro (1)} \qquad \dfrac{\begin{matrix}\Gamma \\ \vdots\, \pi \\ B\end{matrix}}{A \vee B}\ \vee\text{-intro (2)}$	

Remark. Rules are constructors for proofs (not transitions from formulas to formulas)

Logical rules actually are proofs constructors

For instance, the rule $\wedge$-elim$_1$ presented as :

$$\frac{\begin{array}{c}\Gamma \\ \vdots \\ A \wedge B\end{array}}{A}\ \wedge\text{-élim}_1$$

Logical rules actually are proofs constructors

For instance, the rule $\wedge$-elim$_1$ presented as :

$$\dfrac{\begin{matrix}\Gamma \\ \vdots \\ A \wedge B\end{matrix}}{A}\ \wedge\text{-élim}_1$$

is just a shortcut for :

$$\begin{matrix}\Gamma \\ \vdots\ \pi \\ A \wedge B\end{matrix} \quad \longmapsto \quad \left.\dfrac{\begin{matrix}\Gamma \\ \vdots\ \pi \\ A \wedge B\end{matrix}}{A}\ \wedge\text{-elim}_1\right\}\pi'$$

Proof π constructed at Time t

New proof π' constructed at Time $t+1$

Introductions	Eliminations
$\dfrac{\begin{matrix}\Gamma\\ \vdots\,\pi_1\\ A\end{matrix} \qquad \begin{matrix}\Gamma\\ \vdots\,\pi_2\\ B\end{matrix}}{A \wedge B}\ \wedge\text{-intro}$	$\dfrac{\begin{matrix}\Gamma\\ \vdots\,\pi\\ A \wedge B\end{matrix}}{A}\ \wedge\text{-elim (1)}$ $\quad \dfrac{\begin{matrix}\Gamma\\ \vdots\,\pi\\ A \wedge B\end{matrix}}{B}\ \wedge\text{-elim (2)}$
	$\dfrac{\begin{matrix}\Gamma\\ \vdots\,\pi_1\\ A \to B\end{matrix} \qquad \begin{matrix}\Gamma\\ \vdots\,\pi_2\\ A\end{matrix}}{B}\ \to\text{-elim}$
$\dfrac{\begin{matrix}\Gamma\\ \vdots\,\pi\\ A\end{matrix}}{A \vee B}\ \vee\text{-intro (1)}$ $\quad \dfrac{\begin{matrix}\Gamma\\ \vdots\,\pi_3\\ B\end{matrix}}{A \vee B}\ \vee\text{-intro (2)}$	

Rules

Introductions	Eliminations
$\dfrac{\begin{matrix}\Gamma\\ \vdots\ \pi_1\\ A\end{matrix} \qquad \begin{matrix}\Gamma\\ \vdots\ \pi_2\\ B\end{matrix}}{A \wedge B}$ ∧-intro	$\dfrac{\begin{matrix}\Gamma\\ \vdots\ \pi\\ A \wedge B\end{matrix}}{A}$ ∧-elim (1) $\qquad \dfrac{\begin{matrix}\Gamma\\ \vdots\ \pi\\ A \wedge B\end{matrix}}{B}$ ∧-elim (2)
$n\ \dfrac{\begin{matrix}\Gamma\ \ \cancel{A}^{n}\\ \vdots\ \pi\\ B\end{matrix}}{A \to B}$ → -intro	$\dfrac{\begin{matrix}\Gamma\\ \vdots\ \pi_1\\ A \to B\end{matrix} \qquad \begin{matrix}\Gamma\\ \vdots\ \pi_2\\ A\end{matrix}}{B}$ →-elim
$\dfrac{\begin{matrix}\Gamma\\ \vdots\ \pi\\ A\end{matrix}}{A \vee B}$ ∨-intro (1) $\qquad \dfrac{\begin{matrix}\Gamma\\ \vdots\ \pi_3\\ B\end{matrix}}{A \vee B}$ ∨-intro (2)	$n\ \dfrac{\begin{matrix}\Gamma\\ \vdots\ \pi_1\\ A \vee B\end{matrix} \qquad \begin{matrix}\Gamma\ \ \cancel{A}^{n}\\ \vdots\ \pi_2\\ C\end{matrix} \qquad \begin{matrix}\Gamma\ \ \cancel{B}^{n}\\ \vdots\ \pi_3\\ C\end{matrix}}{C}$ ∨-elim

Introductions	Eliminations
$\Gamma \quad \overset{n}{\not{A}}$ $\vdots\ \pi$ $n\ \dfrac{B}{A \to B} \to$ -intro	
	$\Gamma \qquad\qquad \Gamma \quad \overset{n}{\not{A}} \qquad\qquad \Gamma \quad \overset{n}{\not{B}}$ $\vdots\ \pi_1 \qquad\qquad \vdots\ \pi_2 \qquad\qquad \vdots\ \pi_3$ $n\ \dfrac{A \vee B \qquad\qquad C \qquad\qquad C}{C} \vee$-elim

Rules

Introductions	Eliminations
$\begin{array}{c} \Gamma \quad \overset{n}{\not{A}} \\ \vdots\ \pi \\ \end{array}$ $n\ \dfrac{B}{A \to B} \to$ -intro	

Remark 3. Hypothesis desactivation

Rules

Hypothesis "active" at Time t...

$$\Gamma \quad \overbrace{A \; \ldots \; A}^{\text{k times}}$$

$$\vdots\, \pi$$

$$B$$

Proof π constructed at Time t

$$\longmapsto$$

... are desactivated at Time $t+1$

$$\left.\begin{array}{c} \Gamma \quad \overbrace{\not{A}^{\,n} \; \ldots \; \not{A}^{\,n}}^{\text{k times}} \\ \vdots\, \pi \\ {}^{n}\dfrac{B}{A \to B} \to\text{-intro} \end{array}\right\} \pi'$$

New proof π' constructed at Time $t+1$

Initiating proofs construction : the smallest proof tree

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- For any formula A, this one node tree below is a proof :

$$A$$

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- In that tree, the formula A is both :
  - a leaf: so it is a proof under hypothesis A
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- In that tree, the formula A is both :
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- So is a proof of A under hypothesis A.
- Terminology : "identity axiom"

A proof in Natural Deduction : an example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{{}^{3}\,\cancel{X \to (Y \to Z)} \qquad {}^{1}\,\cancel{X}}{Y \to Z}\ \to\text{-élim} \qquad \dfrac{{}^{2}\,\cancel{X \to Y} \qquad {}^{1}\,\cancel{X}}{Y}\ \to\text{-elim}}{Z}\ \to\text{-elim}}{X \to Z}\ \to\text{-intro}\ (1)}{(X \to Y) \to (X \to Z)}\ \to\text{-intro}\ (2)}{(X \to (Y \to Z)) \to ((X \to Y) \to (X \to Z))}\ \to\text{-intro}\ (3)
$$

PART 2

The computational point of view on proofs

An example of non analytical proof

$$
{\scriptstyle 2}\ \dfrac{{\scriptstyle 1}\ \dfrac{\dfrac{\dfrac{\overset{2}{\not{X}} \qquad \overset{1}{\not{Y}}}{X \wedge Y}\ \wedge\text{-intro}}{X}\ \wedge\text{-elim (1)}}{Y \rightarrow X}\ \rightarrow\text{-intro}}{X \rightarrow (Y \rightarrow X)}\ \rightarrow\text{-intro}
$$

An example of non analytical proof

$$\dfrac{\dfrac{\dfrac{\dfrac{[X]^2 \quad [Y]^1}{X \wedge Y}\ \wedge\text{-intro}}{X}\ \wedge\text{-elim (1)}}{Y \to X}\ {}^1 \to\text{-intro}}{X \to (Y \to X)}\ {}^2 \to\text{-intro}$$

- The connective $\wedge$ is not present in $X \to (Y \to X)$ (the proved theorem)

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\dfrac{\dfrac{\dfrac{\dfrac{[X]^{2} \qquad [Y]^{1}}{X \wedge Y}\ \wedge\text{-intro}}{X}\ \wedge\text{-elim (1)}}{Y \to X}\ {\to}\text{-intro}\ (1)}{X \to (Y \to X)}\ {\to}\text{-intro}\ (2)
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- $\wedge$ is an extrinsic element : it cannot be obtained by analyzing the theorem (nor the hypothesis)

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- The connective $\wedge$ is not present in $X \to (Y \to X)$ (the proved theorem)
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- The proof is **not analytical** (it contains an extrinsic element)

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$$\cfrac{2\ \cfrac{1\ \cfrac{\cfrac{\overset{2}{\not X} \qquad \overset{1}{\not Y}}{X \wedge Y}\ \wedge\text{-intro}}{X}\ \wedge\text{-elim (1)}}{Y \to X}\ \to\text{-intro}}{X \to (Y \to X)}\ \to\text{-intro}$$

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Importance of analyticity, w.r.t. heuristics (proof-search): knowing the statement we want to prove, we know in advance the **finite** list of formulas that could appear in the proof we are looking for (namely the subformulas of the statement supposed to be proved).

Back to our example of non analytical proof

$$
{}^{2}\!\!\quad \dfrac{\dfrac{\dfrac{\dfrac{{}^{2}\cancel{X} \qquad {}^{1}\cancel{Y}}{X \wedge Y}\ \wedge\text{-intro}}{X}\ \wedge\text{-elim (1)}}{Y \to X}\ {\to}\text{-intro}\ (1)}{X \to (Y \to X)}\ {\to}\text{-intro}\ (2)
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- In that proof : scarcely introduced, the extrinsic element $\wedge$ happens to be eliminated

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$$\cfrac{\cfrac{\cfrac{\cfrac{[X]^{2} \qquad [Y]^{1}}{X \wedge Y}\ \wedge\text{-intro}}{X}\ \wedge\text{-elim (1)}}{Y \to X}\ {}^{1}\ {\to}\text{-intro}}{X \to (Y \to X)}\ {}^{2}\ {\to}\text{-intro}$$

} a “cut” (alias a “redex”)

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- Gentzen proved:
  - The situation above is general: **Proofs with no** cut **are analytical** (NB : wrong in second order logic)

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- Gentzen proved:
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  - **Analytizability of proofs** : one can transform any proof, in a cut-free (hence analytical) proof of the same theorem.

Gentzen's algorithm for cut elimination: one step

$$
\dfrac{
  \dfrac{
    \begin{array}{c}
      \Gamma_1 \quad [A]^{n}_{\langle 0 \rangle} \; \ldots \; [A]^{n}_{\langle k \rangle} \\
      \vdots\ \pi_1 \\
      B
    \end{array}
  }{A \rightarrow B}\ n\ \rightarrow\text{-intro}
  \qquad
  \begin{array}{c}
    \Gamma_2 \\
    \vdots\ \pi_2 \\
    A
  \end{array}
}{B}\ \rightarrow\text{-elim}
$$

$$\vdots$$

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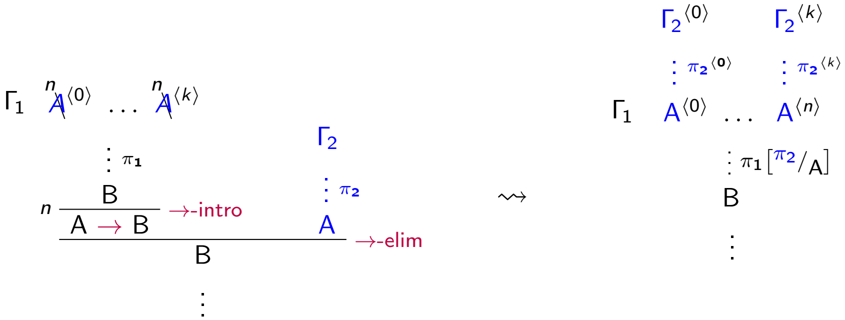

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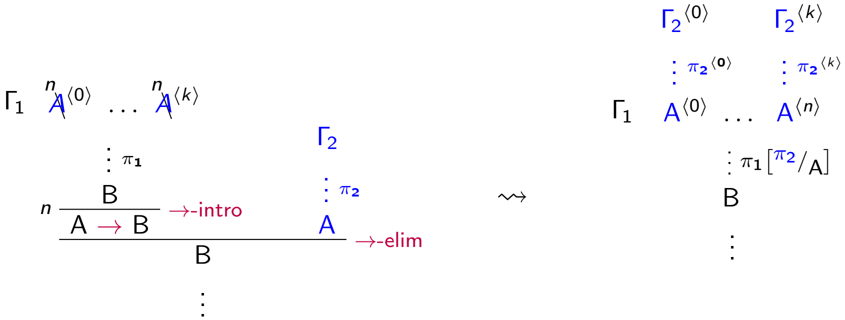

- Complexity ⇔ Duplication ⇔ Multiple occurrences hypothesis

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cut-elimination is the corner stone of Logic.

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  - Lambda calculus (Church, 1934) : a functional ontology; a programming language
  - Curry-Howard isomorphism (1969) :

(Typed Lambda calculus, execution)
=
(Natural Deduction, cut-elimination)

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A corollary of cut-eliminability : Consistency results

For Proof theory (and in particular Linear Logic)
cut-elimination is the corner stone of Logic.

- **Computing theory reasons:**
  - Lambda calculus (Church, 1934) : a functional ontology; a programming language
  - Curry-Howard isomorphism (1969) :

(Typed Lambda calculus, execution)
=
(Natural Deduction, cut-elimination)

Cut-elimination is a conceptual bridge between Logic and Computing theory.

Focusing on negation

Introductions	Eliminations
$\Gamma \quad [A]^{n}$ $\vdots\ \pi$ $n\ \dfrac{\bot}{\neg A}$ ¬-intro	$\Gamma \qquad \Gamma$ $\vdots\ \pi_1 \qquad \vdots\ \pi_2$ $\dfrac{\neg A \qquad A}{\bot}$ →-elim
Intuitionistic absurdum (negation)	Classical absurdum (negation)
Γ $\vdots\ \pi$ $\dfrac{\bot}{A}$ efq	$\Gamma \quad [\neg A]^{n}$ $\vdots$ $n\ \dfrac{\bot}{A}$ raa

Terminology and notations for Natural Deduction systems:

- Minimal : NM
- Intuitionistic : $NJ = NM + \text{efq}$
- Classical : $NK = NJ + \text{raa}$

The "symmetries of classical logic" : dual connectives

- Notation : $A \equiv B$ if A and B are provably equivalent in NK
- De Morgan "laws" :

$$\neg\neg A \equiv A$$

$$\neg(A \wedge B) \equiv \neg A \vee \neg B \qquad \neg A \wedge \neg B \equiv \neg(A \vee B)$$

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- "Symmetries of classical logic" at the provability level
- However those symmetries:
  - are not visible in NK at the level of proofs
  - i.e. are almost not reflected in NK rules

The "dissymmetries of classical Natural Deduction"

Introductions	Eliminations
	$\dfrac{A \wedge B}{A}$ $\wedge$-elim (1) $\quad \dfrac{A \wedge B}{B}$ $\wedge$-elim (2)
$\dfrac{A}{A \vee B}$ $\vee$-intro (1) $\quad \dfrac{B}{A \vee B}$ $\vee$-intro (2)	
Intuitionistic absurdum (negation)	**Classical absurdum (negation)**

The "dissymmetries of classical Natural Deduction"

Introductions	Eliminations
$\dfrac{A \qquad B}{A \wedge B}$ $\wedge$-intro	$\dfrac{A \wedge B}{A}$ $\wedge$-elim (1) $\quad \dfrac{A \wedge B}{B}$ $\wedge$-elim (2)
$\dfrac{A}{A \vee B}$ $\vee$-intro (1) $\quad \dfrac{B}{A \vee B}$ $\vee$-intro (2)	$\dfrac{\begin{matrix}\Gamma \\ \vdots\, \pi_1 \\ A \vee B\end{matrix} \qquad \begin{matrix}\Gamma \quad \overset{n}{\cancel{A}} \\ \vdots\, \pi_2 \\ C\end{matrix} \qquad \begin{matrix}\Gamma \quad \overset{n}{\cancel{B}} \\ \vdots\, \pi_2 \\ C\end{matrix}}{C}$ $\vee$-elim (n)
Intuitionistic absurdum (negation)	Classical absurdum (negation)

The "dissymmetries of classical Natural Deduction"

Introductions	Eliminations
$\dfrac{A \qquad B}{A \wedge B}\ \wedge\text{-intro}$	$\dfrac{A \wedge B}{A}\ \wedge\text{-elim (1)} \qquad \dfrac{A \wedge B}{B}\ \wedge\text{-elim (2)}$
$\dfrac{A}{A \vee B}\ \vee\text{-intro (1)} \qquad \dfrac{B}{A \vee B}\ \vee\text{-intro (2)}$	$\dfrac{\begin{matrix}\Gamma \\ \vdots\ \pi_1 \\ A \vee B\end{matrix} \qquad \begin{matrix}\Gamma\ \ [A]^n \\ \vdots\ \pi_2 \\ C\end{matrix} \qquad \begin{matrix}\Gamma\ \ [B]^n \\ \vdots\ \pi_2 \\ C\end{matrix}}{C}\ n\ \vee\text{-elim}$
$\dfrac{\begin{matrix}\Gamma\ \ [A]^n \\ \vdots\ \pi \\ \bot\end{matrix}}{\neg A}\ n\ \neg\text{-intro}$	$\dfrac{\begin{matrix}\Gamma \\ \vdots\ \pi_1 \\ \neg A\end{matrix} \qquad \begin{matrix}\Gamma \\ \vdots\ \pi_2 \\ A\end{matrix}}{\bot}\ \to\text{-elim}$
Intuitionistic absurdum (negation)	Classical absurdum (negation)
$\dfrac{\begin{matrix}\Gamma \\ \vdots\ \pi \\ \bot\end{matrix}}{A}\ \text{efq}$	$\dfrac{\begin{matrix}\Gamma\ \ [\neg A]^n \\ \vdots \\ \bot\end{matrix}}{A}\ n\ \text{raa}$

From Natural Deduction to Sequent calculus

- Two dissymmetries in the Natural Deduction format :
  - Dissymmetry Hypothesis/Conclusion (*many hypothesis* vs *one conclusion*)
  - Proofs are "conclusion oriented" (the grow down : toward conclusion)

From Natural Deduction to Sequent calculus

- Two dissymmetries in the Natural Deduction format :
  - Dissymmetry Hypothesis/Conclusion (*many hypothesis* vs *one conclusion*)
  - Proofs are "conclusion oriented" (the grow down : toward conclusion)

- Two reasons to move from Natural Deduction to Sequent calculus
  - Recover classical symmetries at the level of proofs (rules)
  - Get rid of difficulties when generalizing cut elimination to classical Natural Deduction

Toward Natural Deduction as a sequent derivation system : a comparison

An arithmetical calculation represented via moves from arithmetical expressions to other ones:

$$(3 \times 2) \times (5 \times 3)$$
$$=$$
$$6 \times (5 \times 3)$$
$$=$$
$$6 \times 15$$
$$=$$
$$90$$

Representation of **the same** calculation via moves from arithmetical identities to other ones:

$$(3 \times 2) \times (5 \times 3) = (3 \times 2) \times (5 \times 3)$$
$$\Downarrow$$
$$(3 \times 2) \times (5 \times 3) = 6 \times (5 \times 3)$$
$$\Downarrow$$
$$(3 \times 2) \times (5 \times 3) = 6 \times 15$$
$$\Downarrow$$

Natural Deduction deriving sequents from sequents

Representation of the progression of a proof via steps from formulas to formulas:

$$\cfrac{\cfrac{\cfrac{X \wedge ((Y \wedge Z) \wedge W)}{(Y \wedge Z) \wedge W}\ \wedge\text{-}\mathbf{elim_1}}{Y \wedge Z}\ \wedge\text{-}\mathbf{elim_2}}{(Y \wedge Z) \vee X}\ \vee\text{-}\mathbf{intro_1}$$

Representation of **the same** proof via steps from 'sequents' to 'sequents':

$$\cfrac{\cfrac{\cfrac{X \wedge ((Y \wedge Z) \wedge W) \ \vdash\ X \wedge ((Y \wedge Z) \wedge W)}{X \wedge ((Y \wedge Z) \wedge W) \ \vdash\ (Y \wedge Z) \wedge W}\ \wedge\text{-}\mathbf{elim_1}}{X \wedge ((Y \wedge Z) \wedge W) \ \vdash\ Y \wedge Z}\ \wedge\text{-}\mathbf{elim_2}}{X \wedge ((Y \wedge Z) \wedge W) \ \vdash\ (Y \wedge Z) \vee X}\ \vee\text{-}\mathbf{intro_1}$$

Two different *notations* for the same proof

Natural Deduction deriving sequents from sequents

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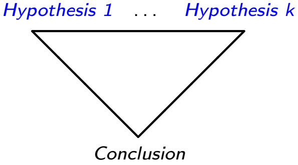
Representation of **the same** proof via steps from 'sequents' to 'sequents':

$$\cfrac{\cfrac{\cfrac{X \wedge ((Y \wedge Z) \wedge W) \ \vdash\ X \wedge ((Y \wedge Z) \wedge W)}{X \wedge ((Y \wedge Z) \wedge W) \ \vdash\ (Y \wedge Z) \wedge W}\ \wedge\text{-elim}_1}{X \wedge ((Y \wedge Z) \wedge W) \ \vdash\ Y \wedge Z}\ \wedge\text{-elim}_2}{X \wedge ((Y \wedge Z) \wedge W) \ \vdash\ (Y \wedge Z) \vee X}\ \vee\text{-intro}_1$$

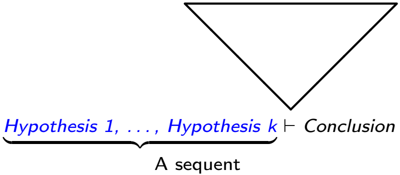
Two different *notations* for the same proof

Deduction of formulas versus Derivation of sequents

A deduction of formulas from formulas



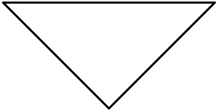
A derivation of sequents from sequents



Deduction of formulas versus Derivation of sequents

A deduction of formulas from formulas

Hypothesis 1 ... *Hypothesis k*



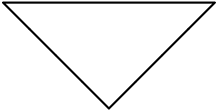
Conclusion

$$\Gamma_1 \quad \overset{n}{\not{A}} \;\ldots\; \overset{n}{\not{A}}$$

$$\vdots\ \pi$$

$$n\ \frac{B}{A \to B}\ \to\text{-intro}$$

A derivation of sequents from sequents



$\underbrace{\textit{Hypothesis 1, \ldots, Hypothesis k}}_{\text{A sequent}} \vdash \textit{Conclusion}$

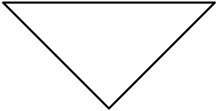
$$\vdots$$

$$\frac{\Gamma, A, \cdots, A \vdash B}{\Gamma \vdash A \to B}\ \to\text{-intro}$$

Deduction of formulas versus Derivation of sequents

A deduction of formulas from formulas

Hypothesis 1 ... *Hypothesis k*



Conclusion

$$\Gamma_1 \quad \overset{n}{\not{A}} \; \dots \; \overset{n}{\not{A}}$$

$$\vdots \; \pi$$

$$n \; \frac{B}{A \; \to \; B} \; \to\text{-intro}$$

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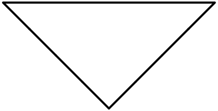


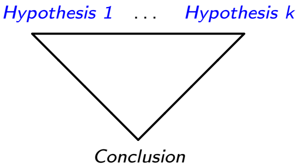
Hypothesis 1, ..., Hypothesis k $\vdash$ *Conclusion*

A sequent

$\vdots$ *(useless)*

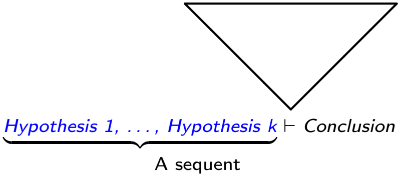
$$\frac{\Gamma, A, \cdots, A \vdash B}{\Gamma \vdash A \to B} \; \to\text{-intro}$$

A deduction of formulas from formulas



$$\dfrac{\begin{array}{c}\Gamma_1 \quad \overset{n}{\not{A}} \;\ldots\; \overset{n}{\not{A}} \\ \vdots\ \pi \\ \mathrm{B}\end{array}}{\mathrm{A} \rightarrow \mathrm{B}}\ n\ \rightarrow\text{-intro}$$

A derivation of sequents from sequents

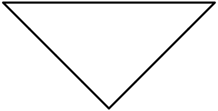


$$\frac{\Gamma, A, \cdots, A \vdash B}{\Gamma \vdash A \rightarrow B}\ \rightarrow\text{-intro}$$

Deduction of formulas versus Derivation of sequents

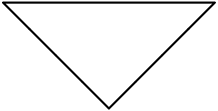
A deduction of formulas from formulas

Hypothesis 1 . . . *Hypothesis k*



Conclusion

A derivation of sequents from sequents



$\underbrace{\textit{Hypothesis 1, \ldots, Hypothesis k}}_{\text{A sequent}} \vdash \textit{Conclusion}$

Hyp

↓

A (Id ax)

↑

Concl

$$\frac{}{A \vdash A}\ \text{Id Axm}$$

Classical Natural Deduction

- **Axiomes identité**

$$\frac{}{\Gamma, A \vdash A}\,\textbf{axm-id}$$

- **Implication**

$$\frac{\Gamma, A, \cdots, A \vdash B}{\Gamma \vdash A \to B}\ \to\textbf{-intro} \qquad \frac{\Gamma \vdash A \to B \qquad \Gamma' \vdash A}{\Gamma, \Gamma' \vdash B}$$

- **Négation**

$$\frac{\Gamma, A, \cdots, A \vdash \bot}{\Gamma \vdash \neg A}\ \neg\textbf{-intro} \qquad \frac{\Gamma \vdash \neg A \qquad \Gamma' \vdash A}{\Gamma, \Gamma' \vdash \bot}\ \neg\textbf{-elim}$$

- **Conjonction**

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge\textbf{-intro} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ \wedge\textbf{-elim 1} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}$$

- **Disjonction**

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \vee\textbf{-i1} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \vee\textbf{-i2} \qquad \frac{\Gamma \vdash A \vee B \qquad \Gamma, A, \cdots, A \vdash C \qquad \Gamma, \ldots}{\Gamma \vdash C}$$

- **L'absurde** $\dfrac{\Gamma \vdash \bot}{\Gamma \vdash A}$ **efq**

$$\frac{\Gamma, \neg A, \cdots, \neg A \vdash \bot}{\Gamma \vdash A}\ \textbf{raa}$$

Classical Natural Deduction : version 1

- **Axiomes identité**

$$\frac{}{\Gamma, A \vdash A}\ \textbf{axm-id}$$

- **Implication**

$$\frac{\Gamma, A, \cdots, A \vdash B}{\Gamma \vdash A \to B}\ \to\textbf{-intro} \qquad \frac{\Gamma \vdash A \to B \qquad \Gamma' \vdash A}{\Gamma, \Gamma' \vdash B}\ -$$

- **Négation**

$$\frac{\Gamma, A, \cdots, A \vdash \bot}{\Gamma \vdash \neg A}\ \neg\textbf{-intro} \qquad \frac{\Gamma \vdash \neg A \qquad \Gamma' \vdash A}{\Gamma, \Gamma' \vdash \bot}\ \neg\textbf{-elim}$$

- **Conjonction**

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge\textbf{-intro} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ \wedge\textbf{-elim 1} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}$$

- **Disjonction**

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \vee\textbf{-i1} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \vee\textbf{-i2} \qquad \frac{\Gamma \vdash A \vee B \qquad \Gamma, A, \cdots, A \vdash C \qquad \Gamma, \ldots}{\Gamma \vdash C}$$

- **L'absurde** $\frac{\Gamma \vdash \bot}{\Gamma \vdash A}$ **efq**

$$\frac{\Gamma, \neg A, \cdots, \neg A \vdash \bot}{\Gamma \vdash A}\ \textbf{raa}$$

- **Axiomes identité**

$$\frac{}{\Gamma, A \vdash A, \Delta}\ \textbf{axm-id}$$

- **Implication**

$$\frac{\Gamma, A, \cdots, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ \to\textbf{-intro} \qquad \frac{\Gamma \vdash A \to B, \Delta \qquad \Gamma' \vdash A, \Delta'}{\Gamma, \Gamma' \vdash B, \Delta, \Delta'}$$

- **Négation**

$$\frac{\Gamma, A, \cdots, A \vdash \bot, \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg\textbf{-intro} \qquad \frac{\Gamma \vdash \neg A, \Delta \qquad \Gamma' \vdash A, \Delta}{\Gamma, \Gamma' \vdash \bot, \Delta}\ \neg\textbf{-elim}$$

- **Conjonction**

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \wedge\textbf{-intro} \qquad \frac{\Gamma \vdash A \wedge B, \Delta}{\Gamma \vdash A, \Delta}\ \wedge\textbf{-elim 1} \qquad \frac{\Gamma \vdash A \wedge B, \Delta}{\Gamma \vdash B, \Delta}$$

- **Disjonction**

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee\textbf{-i1} \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee\textbf{-i2} \qquad \frac{\Gamma \vdash A \vee B, \Delta \qquad \Gamma, A, \cdots, A \vdash C, \Delta \qquad \Gamma, \ldots}{\Gamma \vdash C, \Delta}$$

- **L'absurde** $$\frac{\Gamma \vdash \bot, \Delta}{\Gamma \vdash A, \Delta}\ \textbf{efq}$$

Classical Natural Deduction : version 2

- **Axiomes identité**

$$\frac{}{\Gamma, A \vdash A, \Delta}\ \textbf{axm-id}$$

Sequents $\Gamma \vdash \Delta$ (multi-conclusions, symmetrical)

- **Implication**

$$\frac{\Gamma, A, \cdots, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ \to\textbf{-intro} \qquad \frac{\Gamma \vdash A \to B, \Delta \qquad \Gamma' \vdash A, \Delta'}{\Gamma, \Gamma' \vdash B, \Delta, \Delta'}$$

- **Négation**

$$\frac{\Gamma, A, \cdots, A \vdash \bot, \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg\textbf{-intro} \qquad \frac{\Gamma \vdash \neg A, \Delta \qquad \Gamma' \vdash A, \Delta}{\Gamma, \Gamma' \vdash \bot, \Delta}\ \neg\textbf{-elim}$$

- **Conjonction**

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \wedge\textbf{-intro} \qquad \frac{\Gamma \vdash A \wedge B, \Delta}{\Gamma \vdash A, \Delta}\ \wedge\textbf{-elim 1} \qquad \frac{\Gamma \vdash A \wedge B, \Delta}{\Gamma \vdash B, \Delta}$$

- **Disjonction**

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee\textbf{-i1} \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee\textbf{-i2} \qquad \frac{\Gamma \vdash A \vee B, \Delta \qquad \Gamma, A, \cdots, A \vdash C, \Delta \qquad \Gamma, \ldots}{\Gamma \vdash C, \Delta}$$

- **L'absurde** $$\frac{\Gamma \vdash \bot, \Delta}{\Gamma \vdash A, \Delta}\ \textbf{efq}$$

From Natural Deduction to Sequent Calculus

- The first dissymmetry (Hypothesis/Conclusions) disappeared :

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Sequent calculus : replacing elimination on the right by Introduction on the left

Main qualities of Sequent Calculus

- Symmetries of *classical* theorems (De Morgan) now implemented in rules

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Main qualities of Sequent Calculus

- Symmetries of *classical* theorems (De Morgan) now implemented in rules
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  - Identity Group
  - Logical Group
  - Structural Group
- The distinction between analytic / non analytic proofs is :
  - evident
  - conceptually clear : composition of proofs
- Uncover the dynamical sense of duality : Reversible / Irreversible logical rules

LK - Identity Group and Structural group

Identity group
Identity axiom
$$\text{ax}\ \frac{}{A \vdash A}$$
Cut rule
$$\frac{\Gamma_1 \vdash \Delta_1, A \qquad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \text{cut}$$

LK - Identity Group and Structural group

Identity group

Identity axiom

$$\text{ax}\ \frac{}{A \vdash A}$$

Cut rule

$$\frac{\Gamma_1 \vdash \Delta_1, A \qquad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \text{cut}$$

Structural group

Contractions

$$\frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ \text{ctr} \qquad \text{ctr}\ \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}$$

Weakenings

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ \text{w} \qquad \text{w}\ \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}$$

LK - Identity Group and Structural group

Identity group

Identity axiom

$$\text{ax}\ \frac{}{A \vdash A}$$

Structural group

Contractions

$$\frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ \text{ctr} \qquad \text{ctr}\ \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}$$

Weakenings

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ \text{w} \qquad \text{w}\ \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}$$

LK - Logical group

Logical group : unary connectives (negation)	
Negation	
$\dfrac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg$	$\neg\ \dfrac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}$

LK - Logical group

Logical group : unary connectives (negation)

Negation

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg \qquad\qquad \neg\ \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}$$

Logical group : binary dual connectives

rules for conjunction

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'}\ \wedge \qquad\qquad \wedge\ \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$

rules for disjunction

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee \qquad\qquad \vee\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'}$$

LK - Logical group

Logical group : unary connectives (negation)	
Negation	
$\dfrac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg$	$\neg\ \dfrac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}$

Logical group : binary dual connectives	
rules for conjunction	
$\dfrac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'}\ \wedge$	$\wedge\ \dfrac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$
rules for disjunction	
$\dfrac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee$	$\vee\ \dfrac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'}$

LK - Logical group

Logical group : unary connectives (negation)	
Negation	
$$\dfrac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg$$	$$\neg\ \dfrac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}$$

Logical group : binary dual connectives	
rules for conjunction	
$$\dfrac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'}\ \wedge$$	$$\wedge\ \dfrac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$
rules for disjunction	
$$\dfrac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee$$	$$\vee\ \dfrac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'}$$

LK - Logical group

Logical group : unary connectives (negation)

Negation

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg \qquad\qquad \neg\ \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}$$

Logical group : binary dual connectives

rules for conjunction

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'}\ \wedge \qquad\qquad \wedge\ \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$

rules for disjunction

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee \qquad\qquad \vee\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'}$$

rules for conjunction

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \wedge \qquad\qquad \wedge\ \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \wedge\ \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$

rules for disjunction

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee \qquad\qquad \vee\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\vdash \Gamma, A \vee B \vdash \Delta}$$

LK - Logical group

Logical group : unary connectives (negation)

Negation

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg \qquad\qquad \neg\ \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}$$

Logical group : binary dual connectives

rules for conjunction

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'}\ \wedge \qquad\qquad \wedge\ \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$

rules for disjunction

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee \qquad\qquad \vee\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'}$$

rules for conjunction

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \wedge \qquad\qquad \wedge\ \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \wedge\ \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$

rules for disjunction

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee \qquad\qquad \vee\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\vdash \Gamma, A \vee B \vdash \Delta}$$

LK - Logical group

Logical group : unary connectives (negation)

Negation

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg \qquad\qquad \neg\ \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}$$

Logical group : binary dual connectives

rules for conjunction

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'}\ \wedge \qquad\qquad \wedge\ \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$

rules for disjunction

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee \qquad\qquad \vee\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'}$$

rules for conjunction

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \wedge \qquad \wedge\ \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \wedge\ \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$

rules for disjunction

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee \qquad \vee\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\vdash \Gamma, A \vee B \vdash \Delta}$$

LK - Logical group

Logical group : unary connectives (negation)

Negation

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg \qquad\qquad \neg\ \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}$$

Logical group : binary dual connectives

Multiplicative rules for conjunction

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'}\ \wedge \qquad\qquad \wedge\ \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$

Multiplicative rules for disjunction

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee \qquad\qquad \vee\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'}$$

Additive rules for conjunction

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \wedge \qquad \wedge\ \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \wedge\ \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$

Additive rules for disjunction

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee \qquad \vee\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\vdash \Gamma, A \vee B \vdash \Delta}$$

LK - Logical group

Logical group : unary connectives (negation)

Negation

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\;\neg \qquad\qquad \neg\;\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}$$

Logical group : binary dual connectives

Multiplicative
rules for conjunction

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'}\;\wedge \qquad\qquad \wedge\;\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$

Multiplicative
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$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\;\vee \qquad\qquad \vee\;\frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'}$$

Additive
rules for conjunction

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\;\wedge \qquad\qquad \wedge\;\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \wedge\;\frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$

Additive
rules for disjunction

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\;\vee \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\;\vee \qquad\qquad \vee\;\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\vdash \Gamma, A \vee B \vdash \Delta}$$

LK - Logical group

Logical group : unary connectives (negation)

Negation

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\neg \qquad\qquad \neg\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}$$

Logical group : binary dual connectives

Multiplicative
rules for conjunction

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'}\wedge \qquad\qquad \wedge\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$

Multiplicative
rules for disjunction

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\vee \qquad\qquad \vee\frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'}$$

Additive
rules for conjunction

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\wedge \qquad \wedge\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \wedge\frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$

Additive
rules for disjunction

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\vee \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\vee \qquad \vee\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\vdash \Gamma, A \vee B \vdash \Delta}$$

LK - Logical group

Logical group : unary connectives (negation)

Negation

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg \qquad\qquad \neg\ \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}$$

Logical group : binary dual connectives

Multiplicative
rules for conjunction

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'}\ \wedge \qquad\qquad \wedge\ \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$

Multiplicative
rules for disjunction

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee \qquad\qquad \vee\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'}$$

Additive
rules for conjunction

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \wedge \qquad \wedge\ \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \wedge\ \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$

Additive
rules for disjunction

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee \qquad \vee\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\vdash \Gamma, A \vee B \vdash \Delta}$$

LK - Logical group

Logical group : unary connectives (negation)

Negation

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg \qquad\qquad \neg\ \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}$$

Logical group : binary dual connectives

Multiplicative rules for conjunction

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \overset{m}{\wedge} B, \Delta, \Delta'}\ \overset{m}{\wedge} \qquad\qquad \overset{m}{\wedge}\ \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \overset{m}{\wedge} B \vdash \Delta}$$

Multiplicative rules for disjunction

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \overset{m}{\vee} B, \Delta}\ \overset{m}{\vee} \qquad\qquad \overset{m}{\vee}\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \overset{m}{\vee} B \vdash \Delta, \Delta'}$$

Additive rules for conjunction

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \overset{a}{\wedge} B, \Delta}\ \overset{a}{\wedge} \qquad\qquad \overset{a}{\wedge}\ \frac{\Gamma, A \vdash \Delta}{\Gamma, A \overset{a}{\wedge} B \vdash \Delta} \qquad \overset{a}{\wedge}\ \frac{\Gamma, B \vdash \Delta}{\Gamma, A \overset{a}{\wedge} B \vdash \Delta}$$

Additive rules for disjunction

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \overset{a}{\vee} B, \Delta}\ \overset{a}{\vee} \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \overset{a}{\vee} B, \Delta}\ \overset{a}{\vee} \qquad\qquad \overset{a}{\vee}\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\vdash \Gamma, A \overset{a}{\vee} B \vdash \Delta}$$

PART 3

Classical logic decomposed

Additive and Multiplicative styles equivalent modulo the structural rules

- In presence of the structural rules, the distinction between multiplicative and additive styles degenerates, i.e. one has :

$$A \overset{a}{\vee} B \equiv A \overset{m}{\vee} B$$

$$A \overset{a}{\wedge} B \equiv A \overset{m}{\wedge} B$$

Additive and Multiplicative styles equivalent modulo the structural rules

- In presence of the structural rules, the distinction between multiplicative and additive styles degenerates, i.e. one has :

$$A \overset{a}{\vee} B \equiv A \overset{m}{\vee} B$$

$$A \overset{a}{\wedge} B \equiv A \overset{m}{\wedge} B$$

Indeed (for instance) :

$$\dfrac{\dfrac{\dfrac{A \vdash A}{A \vdash A, B}\,\mathrm{w} \qquad \dfrac{B \vdash B}{B \vdash A, B}\,\mathrm{w}}{A \overset{a}{\vee} B \vdash A\,,\, B}\,\overset{a}{\vee}}{A \overset{a}{\vee} B \vdash A \overset{m}{\vee} B}\,\overset{m}{\vee}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A \overset{m}{\vee} B \vdash A\,,\, B}\,\overset{m}{\vee}}{A \overset{m}{\vee} B \vdash A \overset{a}{\vee} B\,,\, B}\,\overset{a}{\vee}}{A \overset{m}{\vee} B \vdash A \overset{a}{\vee} B\,,\, A \overset{a}{\vee} B}\,\overset{a}{\vee}}{A \overset{m}{\vee} B \vdash A \overset{a}{\vee} B}\,\mathrm{ctr}$$

Additive and Multiplicative styles equivalent modulo the structural rules

- But structural rules are needed for that (easy to show once eliminability of cut is proved).

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- So in the fragment of LK with no structural rules, the rules with the various style define genuine (non equivalent) connectives.

Additive and Multiplicative styles equivalent modulo the structural rules

- But structural rules are needed for that (easy to show once eliminability of cut is proved).
- So in the fragment of LK with no structural rules, the rules with the various style define genuine (non equivalent) connectives.
- Notation and terminology :
  - $\overset{m}{\wedge}$ noted $\otimes$ ("tensor", "times")
  - $\overset{m}{\vee}$ noted $⅋$ ("par", "co-tensor")
  - $\overset{a}{\wedge}$ noted & ("with")
  - $\overset{a}{\vee}$ noted $\oplus$ ("plus")

MALL : Identity group

Identity group
Identity axiom $$\text{ax}\ \frac{}{A \vdash A}$$
Cut rule $$\frac{\Gamma_1 \vdash \Delta_1, A \qquad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \text{cut}$$

Logical group : unary connectives (negation)

Negation

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg \qquad \neg\ \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}$$

Logical group : binary dual connectives

Multiplicative conjunction
($\otimes$, "Tensor")

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'}\ \otimes \qquad \otimes\ \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}$$

Multiplicative disjunction
($\parr$, "Par")

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta}\ \parr \qquad \parr\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \parr B \vdash \Delta, \Delta'}$$

Additive conjunction
(&, "With')

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \,\&\, B, \Delta}\ \& \qquad \&\ \frac{\Gamma, A \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta} \qquad \&\ \frac{\Gamma, B \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta}$$

Additive disjunction
($\oplus$, "Plus')

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta}\ \oplus \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta}\ \oplus \qquad \oplus\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta}$$

Logical group : unary connectives (negation)

Negation

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg \qquad\qquad \neg\ \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}$$

Logical group : binary dual connectives

Multiplicative conjunction
(⊗, "Tensor")

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'}\ \otimes \qquad\qquad \otimes\ \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}$$

Multiplicative disjunction
(⅋, "Par")

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A ⅋ B, \Delta}\ ⅋ \qquad\qquad ⅋\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A ⅋ B \vdash \Delta, \Delta'}$$

Additive conjunction
(&, "With')

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \,\&\, B, \Delta}\ \& \qquad\qquad \&\ \frac{\Gamma, A \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta} \qquad \&\ \frac{\Gamma, B \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta}$$

Additive disjunction
(⊕, "Plus')

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta}\ \oplus \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta}\ \oplus \qquad\qquad \oplus\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta}$$

ALL : Logical group

Logical group : unary connectives (negation)

Negation

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg \qquad\qquad \neg\ \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}$$

Logical group : binary dual connectives

Multiplicative conjunction
(⊗, "Tensor")

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'}\ \otimes \qquad\qquad \otimes\ \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}$$

Multiplicative disjunction
(⅋, "Par")

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A ⅋ B, \Delta}\ ⅋ \qquad\qquad ⅋\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A ⅋ B \vdash \Delta, \Delta'}$$

Additive conjunction
(&, "With')

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \,\&\, B, \Delta}\ \& \qquad\qquad \&\ \frac{\Gamma, A \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta} \qquad \&\ \frac{\Gamma, B \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta}$$

Additive disjunction
(⊕, "Plus')

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta}\ \oplus \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta}\ \oplus \qquad\qquad \oplus\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta}$$

MALL : adding 0-ary connectives (neutrals)

The language is enriched with four 0-ary connectives (thus formulas) :

Multiplicative ones : 1 and $\bot$ Additive ones : 0 and $\top$

MALL : adding 0-ary connectives (neutrals)

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MALL Logical group continued: Neutrals	
0-ary multiplicatives (neutrals)	
$\dfrac{}{\vdash 1}\ \mathbf{1}$	$\bot\ \dfrac{\Gamma \vdash \Delta}{\Gamma, 1 \vdash \Delta}$
$\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \bot}\ \bot$	$\bot\ \dfrac{}{\bot \vdash}$
0-ary additives (neutrals)	
$\dfrac{}{\Gamma \vdash \Delta, \top}\ \top$	*No left intro for* $\top$
No right rule for 0	$\mathbf{0}\ \dfrac{}{\Gamma, 0 \vdash \Delta}$

MALL : adding 0-ary connectives (neutrals)

The language is enriched with four 0-ary connectives (thus formulas) :

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MALL Logical group continued: Neutrals	
0-ary multiplicatives (neutrals)	
$\dfrac{}{\vdash 1}\ \mathbf{1}$	$\bot\ \dfrac{\Gamma \vdash \Delta}{\Gamma, 1 \vdash \Delta}$
$\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \bot}\ \bot$	$\bot\ \dfrac{}{\bot \vdash}$
0-ary additives (neutrals)	
$\dfrac{}{\Gamma \vdash \Delta, \top}\ \top$	*No left intro for* $\top$
No right rule for 0	$\mathbf{0}\ \dfrac{}{\Gamma, 0 \vdash \Delta}$

They are neutrals :

- 1 is provably neutral for $\otimes$ — $\bot$ is provably neutral for $⅋$
- $\top$ is provably neutral for $\&$ — 0 is provably neutral for $\oplus$

MLL, ALL, MALL are computational fragments

Each of MLL, ALL (and thus MALL) satisfies :

- Cut-elimination (no contractions $\Rightarrow$ low complexity process)

MLL, ALL, MALL are computational fragments

Each of MLL, ALL (and thus MALL) satisfies :

- Cut-elimination (no contractions $\Rightarrow$ low complexity process)
- Atomization of axioms, i.e. Identities are canonically provable from atomic initial sequents :

$$\otimes\;\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \otimes B}\;\otimes}{A \otimes B \vdash A \otimes B} \qquad\qquad \dfrac{\parr\;\dfrac{A \vdash A \qquad B \vdash B}{A \parr B \vdash A, B}}{A \parr B \vdash A \parr B}\;\parr$$

$$\dfrac{\&\;\dfrac{A \vdash A}{A \,\&\, B \vdash A} \qquad \&\;\dfrac{B \vdash B}{A \,\&\, B \vdash B}}{A \,\&\, B \vdash A \,\&\, B}\;\& \qquad\qquad \oplus\;\dfrac{\dfrac{A \vdash A}{A \vdash A \oplus B}\;\oplus \qquad \dfrac{B \vdash B}{B \vdash A \oplus B}\;\oplus}{A \oplus B \vdash A \oplus B}$$

$$\mathbf{1}\;\dfrac{\dfrac{}{\vdash 1}\;\mathbf{1}}{1 \vdash 1} \qquad\qquad \dfrac{\bot\;\dfrac{}{\bot \vdash}}{\bot \vdash \bot}\;\bot$$

$$\dfrac{}{\top \vdash \top}\;\top \qquad\qquad \mathbf{0}\;\dfrac{}{0 \vdash 0}$$

De Morgan dualities

- Each fragment MLL, ALL (and thus MALL) satisfies De Morgan equivalences:

$\neg(A \otimes B) \equiv_{\mathsf{MLL}} \neg A \parr \neg B$ $\qquad$ $\neg(A \parr B) \equiv_{\mathsf{MLL}} \neg A \otimes \neg B$

$\neg 1 \equiv_{\mathsf{MLL}} \bot$ $\qquad$ $\neg \bot \equiv_{\mathsf{MLL}} 1$

$\neg(A \,\&\, B) \equiv_{\mathsf{ALL}} \neg A \oplus \neg B$ $\qquad$ $\neg(A \oplus B) \equiv_{\mathsf{ALL}} \neg A \,\&\, \neg B$

$\neg \top \equiv_{\mathsf{ALL}} 0$ $\qquad$ $\neg 0 \equiv_{\mathsf{ALL}} \top$

De Morgan dualities

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  $\neg(A \otimes B) \equiv_{\mathsf{MLL}} \neg A \parr \neg B$ $\qquad$ $\neg(A \parr B) \equiv_{\mathsf{MLL}} \neg A \otimes \neg B$

  $\neg 1 \equiv_{\mathsf{MLL}} \bot$ $\qquad$ $\neg \bot \equiv_{\mathsf{MLL}} 1$

  $\neg(A \,\&\, B) \equiv_{\mathsf{ALL}} \neg A \oplus \neg B$ $\qquad$ $\neg(A \oplus B) \equiv_{\mathsf{ALL}} \neg A \,\&\, \neg B$

  $\neg \top \equiv_{\mathsf{ALL}} 0$ $\qquad$ $\neg 0 \equiv_{\mathsf{ALL}} \top$

- Pairs of mutually dual connectives :

  $\otimes / \parr \qquad \oplus / \& \qquad 1/\bot \qquad 0/\top \qquad (\text{ and } \forall/\exists)$

De Morgan dualities

- Each fragment MLL, ALL (and thus MALL) satisfies De Morgan equivalences:

$$\neg(A \otimes B) \equiv_{\mathsf{MLL}} \neg A \parr \neg B \qquad \neg(A \parr B) \equiv_{\mathsf{MLL}} \neg A \otimes \neg B$$

$$\neg 1 \equiv_{\mathsf{MLL}} \bot \qquad \neg \bot \equiv_{\mathsf{MLL}} 1$$

$$\neg(A \,\&\, B) \equiv_{\mathsf{ALL}} \neg A \oplus \neg B \qquad \neg(A \oplus B) \equiv_{\mathsf{ALL}} \neg A \,\&\, \neg B$$

$$\neg \top \equiv_{\mathsf{ALL}} 0 \qquad \neg 0 \equiv_{\mathsf{ALL}} \top$$

- Pairs of mutually dual connectives :

$$\otimes / \parr \qquad \oplus / \& \qquad 1/\bot \qquad 0/\top \qquad (\text{ and } \forall/\exists)$$

- Symmetry or chattering ? Up to the exchanges left/right and the exchange of dual connectives, everything is said twice. For instance :

$$\otimes\,\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \otimes B}\,\otimes}{A \otimes B \vdash A \otimes B} \qquad\qquad \parr\,\dfrac{A \vdash A \qquad B \vdash B}{\dfrac{A \parr B \vdash A, B}{A \parr B \vdash A \parr B}\,\parr}$$

Ceasing chattering : toward monolateral sequent calculus

- Goal : to divide the number of rules for binary connectives by two.
- Tool: replace negation as a unary function by the binary relation of duality.

Ceasing chattering : toward monolateral sequent calculus

- Goal : to divide the number of rules for binary connectives by two.
- Tool: replace negation as a unary function by the binary relation of duality.
- Step 1: we change the notion of formulas (our old set of formulas will be quotiented by de Morgan equivalences):
  - Negation no more present as a connective, but as a defined operation $(.)^\perp$
  - Atoms come by pairs : each atom X comes with its dual noted $X^\perp$
  - $(X^\perp)^\perp = X$
  - The dual $A^\perp$ of A is "the De Morganized" form of $\neg A$
  - For instance : if $A = (X \otimes (Y^\perp \,\&\, Z))$, then $A^\perp$ *denotes* $(X^\perp \parr (Y \oplus Z^\perp))$

Ceasing chattering : toward monolateral sequent calculus

- Goal : to divide the number of rules for binary connectives by two.
- Tool: replace negation as a unary function by the binary relation of duality.
- Step 1: we change the notion of formulas (our old set of formulas will be quotiented by de Morgan equivalences):
  - Negation no more present as a connective, but as a defined operation $(.)^\perp$
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  - For instance : if $A = (X \otimes (Y^\perp \,\&\, Z))$, then $A^\perp$ *denotes* $(X^\perp \parr (Y \oplus Z^\perp))$
- Step 2 : we fold the left side on the right side up; and, in the "Identity group", we replace identity constraints on formulas (the one on the hypothesis side, the one on the conclusions side) by duality constraints (on the conclusions side)

MALL,MLL, ALL as monolateral systems

Identity group (better called : duality group)

$$\frac{}{\vdash A, A^{\perp}}\ \text{ax}$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\ \text{cut}$$

Logical group (dual connectives)

Multiplicatives

Binary multiplicatives

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\ \otimes \qquad\qquad \frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\ \parr$$

0-ary multiplicatives (neutrals)

$$\frac{}{\vdash 1}\ \mathbf{1} \qquad\qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ \bot$$

Additives

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Gamma}{\vdash A \,\&\, B, \Gamma}\ \& \qquad\qquad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma}\ \oplus \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}\ \oplus$$

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$$\frac{}{\top, \Gamma}\ \top \qquad\qquad \textit{No rule for } 0$$

Toward a reintroduction of structural rules

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- LL answer :
  - introduce "aspect" in the logical language
  - Aspect : a category coming from natural languages grammar
  - In indo-european languages generally implemented by tenses : perfect *vs* imperfect

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- The dual of ? will be noted ! (they are unary connectives or "modalities")

LL (as a monolateral sequent calculus)

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Logical group

Multiplicatives

2-ary multiplicatives

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Exponentials

$$\frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\ ? \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ !$$

Structural group

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- a good way to figure this (and this anticipates the proofnets with exponentials) is to picture the corresponding subproof in a box.
- When one does so, the formula $!A$ is called the main door of the box, the formulas in $?\Gamma$ are called the auxiliary doors of that box.

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- Knowing that contracted or weakened formulas are always prefixed by a ?, only boxes can be so duplicated or erased. So, that the promotion rule is but a "declaration" that the concerned subproof will be potentially subject to "non linear" manipulations (duplications, erasures) during the process.

The dynamic of exponentials (continued)

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  - When the main door $!A$ of a box is confronted via a cut to a formula $?A^{\perp}$ being auxiliary door of another box, the first box will enter in the second one : the dynamic of exponential thus underline a subtile analyse of the modification of the depth at which substitutions happen

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  - When the main door $!A$ of a box is confronted via a cut to a derelicted formula $?A^\perp$, the "declaration" evoked in the previous item is forgotten (dynamically, the dereliction rule is an instruction producing the erasure of the box declaration - but not of the content of the box)

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- Notice that variants of linear logic take opportunity of these decomposition to freeze those possibilities, hence designing sub-logics of LL in which the computation are "tamed" (implicit complexity).

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  - made the additives and the multiplicatives distinction emerge
  - gave, through exponentials, an analyse of the substitution process (focusing on specific non linear manipulations and depth modification mechanisms)
- Initial slogan explained :

Linear Logic is Classical logic, but decomposed and observed through the microscope of the computational point of view on proofs.