Denotational semantics of linear logic

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- Not about formulas (truth) but about the "meaning" of proofs
- Originated in the study of programming (λ-calculus)

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Formula	Type	Object (mathematical space)

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- a notion of morphism $=$ a category
- a denotational model $=$ a category with some additional structure

How linear logic got its name

Models of typed λ-calculus are cartesian closed categories ($\rightarrow$, $\times$).

Quantitative semantics, in a nutshell (Girard, early 80's, *before* LL)

objects = sets   morphisms $\mathrm{QS}(A, B)$ = analytic functions $R^A \rightarrow R^B$

- alternative to Scott domains, motivated by non-determinism
- Analytic functions are defined by power series: there is a family of coefficients $(\varphi_{\bar{a},\beta})_{\bar{a}\in\mathfrak{M}_f(A),b\in B}$ such that

$$\varphi(\alpha)_b = \sum_{\bar{a}\in\mathfrak{M}_f(A)} \varphi_{\bar{a},b}\alpha^{\bar{a}} \qquad \text{where} \qquad \alpha^{[a_1,\dots,a_n]} = \alpha_{a_1}\cdots\alpha_{a_n}$$

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Linear decomposition of the function space

Setting $\alpha^!_{\overline{a}} = \alpha^{\overline{a}}$ we obtain $\alpha^! \in R^{\mathfrak{M}_f(A)}$ and $\varphi(\alpha) = \varphi \cdot \alpha^!$ where $\cdot$ denotes *matrix application*:

$$\mathrm{An}\left(R^A, R^B\right) \cong \mathrm{Lin}\left(R^{\mathfrak{M}_f(A)}, R^B\right)$$

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$$\begin{array}{rcl} \mathrm{An}\left(R^A, R^B\right) & \cong & \mathrm{Lin}\left(R^{\mathfrak{M}_f(A)}, R^B\right) \\ A \Rightarrow B & \cong & !A \multimap B \end{array}$$

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A category of finite dimensional vector spaces: Mat

objects = finite sets   morphisms $\mathrm{Mat}(A, B)$ = matrices $R^{A\times B}$

- The finite set A is the canonical base of R^A
- Matrices $R^{A\times B}$ represent linear maps from R^A to R^B

Proofs as matrices

We interpret formulas A as finite sets $[\![A]\!]$ and proofs $\pi : A \vdash B$ as matrices $\mathrm{Mat}([\![A]\!], [\![B]\!])$: for all $a \in [\![A]\!]$ and $b \in [\![B]\!]$, we have $[\![\pi]\!]_{a,b} \in R$.

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- The axiom is the identity matrix

$$\left[\!\!\left[\frac{}{\vdash A^{\perp}, A}\ \mathsf{ax} \right]\!\!\right]_{a,a'} = \delta_{a,a'}$$

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- Cut is composition, *i.e.* matrix product:

$$\left[\!\!\left[\frac{\begin{array}{c}\pi \\ \vdots \\ \vdash A^\perp, B\end{array} \qquad \begin{array}{c}\pi' \\ \vdots \\ \vdash B^\perp, C\end{array}}{\vdash A^\perp, C}\ \mathsf{cut} \right]\!\!\right]_{a,c} = \sum_{b \in [\![B]\!]} [\![\pi]\!]_{a,b} [\![\pi']\!]_{b,c}$$

The symmetric monoidal closed structure: $\otimes$, $\multimap$

Tensor product

Given matrices $\varphi \in R^{A\times B}$ and $\varphi' \in R^{A'\times B'}$, we can form their tensor product $\varphi \otimes \varphi' \in R^{(A\times A')\times(B\times B')}$ setting $(\varphi \otimes \varphi')_{(a,a'),(b,b')} = \varphi_{a,b}\varphi'_{a',b'}$

- Set $[\![A \otimes B]\!] = [\![A]\!] \times [\![B]\!]$
- This is associative and commutative (up to isomorphism)
- There is a unit $\mathbf{1} = \{*\}$

Space of linear maps

$A \times B$ is also the "object of morphisms" from A to B:

- Set $[\![A \multimap B]\!] = [\![A]\!] \times [\![B]\!]$
- We have an adjunction: $\mathrm{Mat}(A \otimes B, C) \cong \mathrm{Mat}(A, B \multimap C)$

The $*$-autonomous structure: $A^\perp = A \multimap \perp$

Dual space

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- Fix $[\![\perp]\!]$ to be a singleton: $[\![\perp]\!] = \{*\}$ so that $\mathrm{Lin}(R^A, R) \cong \mathrm{Mat}(A, \perp)$
- By monoidal closedness, we can derive a canonical morphism $\varphi \in \mathrm{Mat}(A, (A \multimap \perp) \multimap \perp)$ from the identity on $A \multimap \perp$: $\varphi_{a,((a',*),*)} = \delta_{a,a'}$
- φ is an isomorphism: this is $*$-autonomy

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- This isomorphism reflects the identity $A^{\perp\perp} = A$
- One recovers $A \parr B$ as $\left(A^\perp \otimes B^\perp\right)^\perp$
- In fact here $R^{A \times \{*\}} \cong R^A$: we can set, more simply, $[\![A^\perp]\!] = [\![A]\!]$
- In particular $A \otimes B = A \parr B$: the model is compact closed

Interpretation of MLL

General guidelines

- Interpret a proof of $A_1, \ldots, A_n \vdash B$ as a morphism:

$$A_1 \otimes \cdots \otimes A_n \multimap B$$

- By the $*$-autonomous structure, this is equivalent to a morphism

$$\mathbf{1} \multimap A_1^\perp \parr \cdots \parr A_n^\perp \parr B$$

- This reflects the equiprovability of

$$\Gamma \vdash \Delta, \qquad \vdash \Gamma^\perp, \Delta \qquad \text{and} \qquad \Gamma, \Delta^\perp \vdash$$

In Mat

If π is a proof of $\vdash A_1, \ldots, A_n$ then

$$[\![\pi]\!] \in R^{[\![A_1]\!] \times \cdots \times [\![A_n]\!]}$$

$$\left[\!\!\left[\frac{}{\vdash A^\perp, A}\,\mathrm{ax} \right]\!\!\right]_{a,a'} = \delta_{a,a'}$$

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Additives

Cartesian product

- Vector spaces admit cartesian products: $R^A \times R^B \cong R^{A+B}$ where $A + B$ denotes the disjoint union $(\{1\} \times A) \cup (\{2\} \times B)$
- $[\![A \,\&\, B]\!] = [\![A]\!] + [\![B]\!]$
- This is the product type of programming languages: given two morphisms $\varphi \in \mathrm{Mat}(C, A)$ and $\varphi \in \mathrm{Mat}(C, B)$, one forms $\langle \varphi, \psi \rangle \in \mathrm{Mat}(C, A + B)$, hence the rule

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This is the *additive* conjunction
(*Note that $A \otimes B$ is not the type of pairs*)

Additives

Direct sums

- By duality we obtain $A \oplus B = \left(A^{\perp} \mathbin{\&} B^{\perp}\right)^{\perp}$
- Here $[\![A \oplus B]\!] = [\![A \mathbin{\&} B]\!]$ (because of compact closedness)
- This is the sum type of programming languages: it comes with injections $\iota_l \in \mathrm{Mat}(A, A+B)$ and $\iota_r \in \mathrm{Mat}(B, A+B)$ hence the rules

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ \oplus_l \quad \text{and} \quad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ \oplus_r$$

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The *additive* disjunction

In fact we gave only half of the picture and we miss:

- projections for the products
- case definitions for sums

but they follow from pairs and injections by duality (see the cut elimination later)

$$\left[\!\!\left[\dfrac{\begin{array}{cc}\pi_1 & \pi_2 \\ \vdots & \vdots \\ \vdash \Gamma, A & \vdash \Gamma, B\end{array}}{\vdash \Gamma, A \,\&\, B}\ \& \right]\!\!\right]_{s,(i,c)} = \begin{cases} [\![\pi_1]\!]_{s,a} & \text{if } (i,c) = (1,a) \\ [\![\pi_2]\!]_{s,b} & \text{if } (i,c) = (2,b) \end{cases}$$

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Additives

Cut elimination

$$
\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots \\ \vdash \Gamma, A\end{matrix} \qquad \begin{matrix}\pi_2\\ \vdots \\ \vdash \Gamma, B\end{matrix}}{\vdash \Gamma, A \,\&\, B}\,\& \qquad \dfrac{\begin{matrix}\pi'\\ \vdots \\ \vdash \Delta, A^{\perp}\end{matrix}}{\vdash \Delta, A^{\perp} \oplus B^{\perp}}\,\oplus_l}{\vdash \Gamma, \Delta}\,\mathsf{cut}
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- non-local !
- a good notion of proof nets with additives is difficult to design
- but additives live very naturally in the semantics (like units)

The relational model

Rel

Setting $R = \mathbb{B}$ (booleans), we recover the model presented yesterday:

- write $s \in [\![\pi]\!]$ for $[\![\pi]\!]_s = 1$
- a matrix $\varphi \in \mathbb{B}^{A \times B}$ is a relation $\varphi \subseteq A \times B$
- we no longer care about finiteness, since the composition of relations $\varphi \subseteq A \times B$ and $\psi \subseteq B \times C$ is always defined:

$$(a, c) \in \psi\varphi \quad \text{iff} \quad \text{there exists } b \in B \text{ such that } (a, b) \in \varphi \text{ and } (b, c) \in \psi$$

- write Rel for the category of sets and relations

Recall from yesterday that the relational model can be described by a kind of type system:

$$\begin{array}{c} \pi \\ \vdots \\ \vdash \overset{a_1}{A_1}, \ldots, \overset{a_n}{A_n} \end{array} \quad \text{is derivable} \qquad \text{iff} \qquad (a_1, \ldots, a_n) \in \left[\!\!\left[\begin{array}{c} \pi \\ \vdots \\ \vdash A_1, \ldots, A_n \end{array} \right]\!\!\right]$$

$$[\![A \otimes B]\!] = [\![A \parr B]\!] = [\![A]\!] \times [\![B]\!]$$

$$\dfrac{}{\vdash \overset{a}{A^{\perp}}, \overset{a}{A}}\ \text{ax} \qquad \dfrac{\vdash \overset{s}{\Gamma}, \overset{a}{A^{\perp}} \qquad \vdash \overset{a}{A}, \overset{t}{\Delta}}{\vdash \overset{s}{\Gamma}, \overset{t}{\Delta}}\ \text{cut} \qquad \dfrac{\vdash \overset{s}{\Gamma}, \overset{a}{A} \qquad \vdash \overset{b}{B}, \overset{t}{\Delta}}{\vdash \overset{s}{\Gamma}, \overset{(a,b)}{A \otimes B}, \overset{t}{\Delta}}\ \otimes \qquad \dfrac{\vdash \overset{s}{\Gamma}, \overset{a}{A}, \overset{b}{B}}{\vdash \overset{s}{\Gamma}, \overset{(a,b)}{A \parr B}}\ \parr$$

The relational model

Rules for ~~MLL~~ MALL

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$$[\![A \mathbin{\&} B]\!] = [\![A \oplus B]\!] = [\![A]\!] + [\![B]\!]$$

$$\dfrac{\vdash \overset{s}{\Gamma}, \overset{a}{A} \qquad \vdash \Gamma, B}{\vdash \overset{s}{\Gamma}, \overset{(1,a)}{A \mathbin{\&} B}}\ \& \qquad \dfrac{\vdash \Gamma, A \qquad \vdash \overset{s}{\Gamma}, \overset{b}{B}}{\vdash \overset{s}{\Gamma}, \overset{(2,b)}{A \mathbin{\&} B}}\ \& \qquad \dfrac{\vdash \overset{s}{\Gamma}, \overset{a}{A}}{\vdash \overset{s}{\Gamma}, \overset{(1,a)}{A \oplus B}}\ \oplus_l \qquad \dfrac{\vdash \overset{s}{\Gamma}, \overset{b}{B}}{\vdash \overset{s}{\Gamma}, \overset{(2,b)}{A \oplus B}}\ \oplus_r$$

Recall from quantitative semantics: $[\![!A]\!] = \mathfrak{M}_f([\![A]\!])$.

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Exponentials in Rel

Ingredients

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- This has the structure of a comonad with

$$\mathrm{der} = \{([a], a);\ a \in A\} \in \mathrm{Rel}(!A, A)$$

and

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This *(plus a bunch of other conditions to make everything work together)* makes Rel a model of LL.

Structural rules: resource management

$$\dfrac{\vdash \overset{s}{\Gamma}, \overset{a}{A}}{\vdash \overset{s}{\Gamma}, \overset{[a]}{?A}}\ ?d \qquad \dfrac{\vdash \overset{s}{\Gamma}}{\vdash \overset{s}{\Gamma}, \overset{[]}{?A}}\ ?w \qquad \dfrac{\vdash \overset{s}{\Gamma}, \overset{\overline{a}}{?A}, \overset{\overline{a}'}{?A}}{\vdash \overset{s}{\Gamma}, \overset{\overline{a}+\overline{a}'}{?A}}\ ?c$$

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Promotion: resource factory

$$\dfrac{\vdash \overset{\overline{s}_i}{?\Gamma}, \overset{a_i}{A} \quad (\text{for } 1 \leq i \leq n)}{\vdash \overset{\sum_{i=1}^{n} \overline{s}_i}{?\Gamma}, \overset{[a_1,\ldots,a_n]}{!A}}\ !$$

(Quantitative semantics relies on quite complicated formulas for the coefficients.)

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,\mathrm{ax}}{\vdash A^\perp \oplus B^\perp, A}\,\oplus_l}{\vdash ?(A^\perp \oplus B^\perp), A}\,?d}{\vdash ?(A^\perp \oplus B^\perp),\ !A}\,! \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash B^\perp, B}\,\mathrm{ax}}{\vdash A^\perp \oplus B^\perp, B}\,\oplus_r}{\vdash ?(A^\perp \oplus B^\perp), B}\,?d}{\vdash ?(A^\perp \oplus B^\perp),\ !B}\,!}{\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp),\ !A \otimes !B}\,\otimes}{\vdash ?(A^\perp \oplus B^\perp),\ !A \otimes !B}\,?c
$$

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}{\vdash ?(A^{\perp} \oplus B^{\perp}),\ !A}\ !
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\dfrac{\dfrac{}{\vdash \overset{b}{B^{\perp}}, \overset{b}{B}}\ \mathrm{ax}}{\vdash A^{\perp} \oplus B^{\perp}, B}\ \oplus_r
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}{\vdash ?(A^{\perp} \oplus B^{\perp}), ?(A^{\perp} \oplus B^{\perp}),\ !A \otimes !B}\ \otimes
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$$

Example

$$\dfrac{}{\vdash \overset{a}{A^\perp}, \overset{a}{A}}\ \mathrm{ax}$$

$$\dfrac{\vdash A^\perp, A}{\vdash \overset{(1,a)}{A^\perp \oplus B^\perp}, \overset{a}{A}}\ \oplus_l$$

$$\dfrac{\vdash A^\perp \oplus B^\perp, A}{\vdash ?(A^\perp \oplus B^\perp), A}\ ?d$$

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$$\dfrac{}{\vdash \overset{b}{B^\perp}, \overset{b}{B}}\ \mathrm{ax}$$

$$\dfrac{\vdash B^\perp, B}{\vdash \overset{(2,b)}{A^\perp \oplus B^\perp}, \overset{b}{B}}\ \oplus_r$$

$$\dfrac{\vdash A^\perp \oplus B^\perp, B}{\vdash ?(A^\perp \oplus B^\perp), B}\ ?d$$

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$$\dfrac{\vdash ?(A^\perp \oplus B^\perp), \ !A \qquad \vdash ?(A^\perp \oplus B^\perp), \ !B}{\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), \ !A \otimes !B}\ \otimes$$

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Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \overset{a}{A^\perp}, \overset{a}{A}}\;\text{ax}}{\vdash \overset{(1,a\;)}{A^\perp \oplus B^\perp}, \overset{a}{A}}\;\oplus_l}{\vdash \overset{[(1,a\;)]}{?(A^\perp \oplus B^\perp)}, \overset{a}{A}}\;?d}{\vdash\; ?(A^\perp \oplus B^\perp)\,,\quad !A}\;!
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \overset{b}{B^\perp}, \overset{b}{B}}\;\text{ax}}{\vdash \overset{(2,b\;)}{A^\perp \oplus B^\perp}, \overset{b}{B}}\;\oplus_r}{\vdash \overset{[(2,b\;)]}{?(A^\perp \oplus B^\perp)}, \overset{b}{B}}\;?d}{\vdash\; ?(A^\perp \oplus B^\perp)\,,\quad !B}\;!
$$

$$
\dfrac{\dfrac{\vdash\; ?(A^\perp \oplus B^\perp)\,,\quad !A \qquad \vdash\; ?(A^\perp \oplus B^\perp)\,,\quad !B}{\vdash\; ?(A^\perp \oplus B^\perp)\,,\; ?(A^\perp \oplus B^\perp)\,,\qquad !A \otimes !B}\;\otimes}{\vdash \qquad ?(A^\perp \oplus B^\perp) \qquad , \qquad !A \otimes !B}\;?c
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Example

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \overset{a_i}{A^\perp}, \overset{a_i}{A}}\ \mathrm{ax}}{\vdash \overset{(1,a_i)}{A^\perp \oplus B^\perp}, \overset{a_i}{A}}\ \oplus_l}{\vdash \overset{[(1,a_i)]}{?(A^\perp \oplus B^\perp)}, \overset{a_i}{A}}\ ?d}{\vdash \overset{[(1,a_1),\dots,(1,a_n)]}{?(A^\perp \oplus B^\perp)},\ \overset{[a_1,\dots,a_n]}{!A}}\ !$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \overset{b_i}{B^\perp}, \overset{b_i}{B}}\ \mathrm{ax}}{\vdash \overset{(2,b_i)}{A^\perp \oplus B^\perp}, \overset{b_i}{B}}\ \oplus_r}{\vdash \overset{[(2,b_i)]}{?(A^\perp \oplus B^\perp)}, \overset{b_i}{B}}\ ?d}{\vdash \overset{[(2,b_2),\dots,(2,b_p)]}{?(A^\perp \oplus B^\perp)},\ \overset{[b_2,\dots,b_p]}{!B}}\ !$$

$$\dfrac{\dfrac{\vdash ?(A^\perp \oplus B^\perp),\ !A \qquad \vdash ?(A^\perp \oplus B^\perp),\ !B}{\vdash ?(A^\perp \oplus B^\perp),\ ?(A^\perp \oplus B^\perp),\ !A \otimes !B}\ \otimes}{\vdash ?(A^\perp \oplus B^\perp),\ !A \otimes !B}\ ?c$$

Example

$$\dfrac{}{\vdash \underbrace{A^\perp}_{a_i}, \underbrace{A}_{a_i}}\ \mathrm{ax}
\qquad
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\qquad
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$$\dfrac{\vdash A^\perp \oplus B^\perp, A}{\vdash \underbrace{?(A^\perp \oplus B^\perp)}_{[(1,a_i)]}, \underbrace{A}_{a_i}}\ ?d
\qquad
\dfrac{\vdash A^\perp \oplus B^\perp, B}{\vdash \underbrace{?(A^\perp \oplus B^\perp)}_{[(2,b_i)]}, \underbrace{B}_{b_i}}\ ?d$$

$$\dfrac{\vdash ?(A^\perp \oplus B^\perp), A}{\vdash \underbrace{?(A^\perp \oplus B^\perp)}_{[(1,a_1),\dots,(1,a_n)]}, \underbrace{!A}_{[a_1,\dots,a_n]}}\ !
\qquad
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Uniformity

- The relational model is highly non uniform: the union of arbitrary relations is still a relation.
- This is good for capturing non-deterministic superpositions of proofs.
- But it comes with many degeneracies: most notably $A^{\perp} = A$; and it does not see vicious cycles in proof structures.

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- But it comes with many degeneracies: most notably $A^{\perp} = A$; and it does not see vicious cycles in proof structures.

Coherence spaces

A coherence space A is the data of a set $|A|$ (the web of A) and of a reflexive and symmetric relation $\frown_A$ on $|A|$.

The intuitive meaning of $a \frown_A a'$ is that a and a' can appear simultaneously in the semantics of the same proof of A: we will interpret proofs by cliques.

$$\mathfrak{C}(A) = \{\alpha \subseteq |A|\,;\ a \frown_A a',\ \text{for all } a, a' \in \alpha\}$$

Duality

Negation reverses the coherence (but we still need a symmetric relation).
Define three new relations on $|A|$:

- $a \smile_A$ iff $a \not\coh_A a'$;
- $a \asymp_A$ iff $a = a'$ or $a \smile_A a'$;
- $a \frown_A$ iff $a \coh_A a'$ and $a \neq a'$.

Dual

Define $A^\perp$ by $\left|A^\perp\right| = |A|$ and $\coh_{A^\perp} = \asymp_A$.

Multiplicatives

- It is natural to set:

$$\begin{array}{rcl} |A \otimes B| & = & |A| \times |B| \\ (a,b) \frown\!\!\!\!\smile_{A\otimes B} (a',b') & \iff & a \frown\!\!\!\!\smile_A a' \text{ and } b \frown\!\!\!\!\smile_B b' \end{array}$$

- We deduce $\parr$ by de Morgan rules:

$$(a,b) \frown_{A \parr B} (a',b') \iff a \frown_A a' \text{ or } b \frown_B b'$$

- Linear implication $A \multimap B = (A \otimes B^{\perp})^{\perp}$ follows:

$$(a,b) \frown_{A \multimap B} (a',b') \iff a \frown_A a' \text{ entails } b \frown_B b'$$

It should be clear that if $\varphi \in \mathfrak{C}(A \multimap B)$ and $\psi \in \mathfrak{C}(B \multimap C)$, then $\psi\varphi \in \mathfrak{C}(A \multimap C)$ (where composition is that of relations).

A $*$-autonomous category of coherence spaces

Coherence spaces and coherent relations

Write Coh for the category of coherence spaces, with morphisms $\mathrm{Coh}(A, B) = \mathfrak{C}(A \multimap B)$.

- Setting $|\bot| = \{*\}$ equipped with the only possible coherence relation it is clear that $A \multimap \bot \cong A^\bot$.
- We no longer have $A \cong A^\bot$.
- The morphisms giving the $*$-autonomous structure are exactly the same as in Rel.

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Lemma

If we fix an interpretation $[\![X]\!]_{\mathrm{Coh}}$ *of atoms as coherence spaces and consider the relational model where* $[\![X]\!]_{\mathrm{Rel}} = |[\![X]\!]_{\mathrm{Coh}}|$, *then* $[\![\pi]\!]_{\mathrm{Rel}} \in \mathfrak{C}([\![\Gamma]\!]_{\mathrm{Coh}})$ *for all* MLL *proof* π *of* Γ.

Additives

The previous result extends to additives: set $|A \,\&\, B| = |A \oplus B| = |A| + |B|$ and

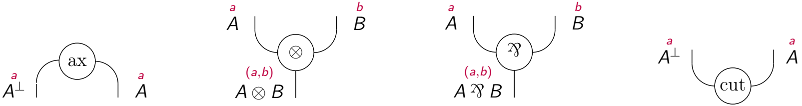
$$(i,c) \coh_{A\&B} (i',c') \iff \begin{cases} & i = i' = 1 \text{ and } c \coh_A c' \\ or & i = i' = 2 \text{ and } c \coh_B c' \\ or & i \neq i' \end{cases}$$

$$(i,c) \coh_{A\oplus B} (i',c') \iff \begin{cases} & i = i' = 1 \text{ and } c \coh_A c' \\ or & i = i' = 2 \text{ and } c \coh_B c' \end{cases}$$

Now $A \,\&\, B \ncong A \oplus B$ and cliques correspond with our intuition:

- a clique in $A \,\&\, B$ corresponds with a pair of cliques in A and B respectively;
- a clique in $A \oplus B$ corresponds with either a clique in A or a clique in B.

- We can play the same game as with Rel:



Experiments in MLL proof nets

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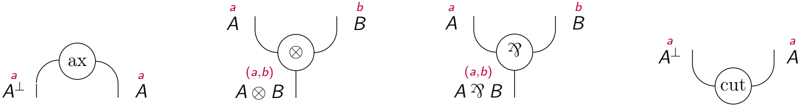
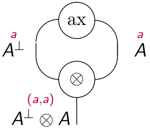


- But now we can detect vicious cycles:



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Experiments in MLL proof nets

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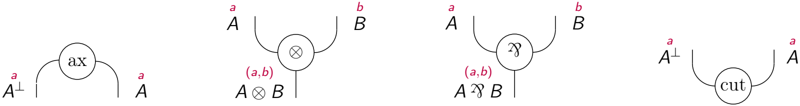

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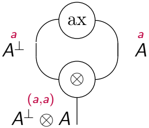

$(a, a) \smile_{A^\perp \otimes A} (a', a')$ as soon as $a \neq a' \in |A|$

In fact one can show that (for a well chosen interpretation of atoms) a cut-free MLL structure is correct iff its relational semantics is clique.

Exponentials

Multiset based

- A natural choice is $|!A| = \mathfrak{M}_f(|A|)$.
- Set $[a_1, \ldots, a_n] \frown_{!A} \left[a'_1, \ldots, a'_p\right]$ iff $a_i \frown_A a'_j$ for all i and j.
- But this is not reflexive!

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Multicliques

- We set $|!A|$ to be the set of all finite multicliques (finite multisets whose support is a clique).
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- $|?A|$ is the set of multicliques in $A^\perp$
- $\frown_{?A}$ is a strange beast.

The system of annotations now comes with side conditions:

$$\frac{\vdash \overset{s}{\Gamma}, \overset{a}{A}}{\vdash \overset{s}{\Gamma}, \overset{[a]}{?A}}\ ?d \qquad \frac{\vdash \overset{s}{\Gamma}}{\vdash \overset{s}{\Gamma}, \overset{[]}{?A}}\ ?w \qquad \frac{\vdash \overset{s}{\Gamma}, \overset{\bar{a}}{?A}, \overset{\bar{a}'}{?A} \quad \bar{a} + \bar{a}' \in |?A|}{\overset{s}{\Gamma}, \overset{\bar{a}+\bar{a}'}{?A}}\ ?c$$

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Coherence spaces were first introduced as a refinement of qualitative domains: particular sets of subsets.

Cliques

- We set $|!A|$ to be the set of all finite cliques of A.
- Then $\tilde{a} \frown \tilde{a}'$ iff $\tilde{a} \cup \tilde{a}' \in |!A|$.

- $?A$ is a still strange beast.
- And we still need annotations.
- This model is less fine but it has a close relationship with the notion of stable functions which was crucial in the invention of coherence spaces.

Orthogonality

Coherence spaces (at least for MLL) are an instance of a general construction:

- The core of the interpretation is the $*$-autonomous structure: composition is the interaction between a proof $\vdash \Gamma, A$ and a counterproof $\vdash A^{\perp}, \Delta$.
- We can refine a preexisting model by selecting morphisms that interact "nicely".

Orthogonality

Coherence spaces as spaces of cliques

- Observe that if $\alpha \in \mathfrak{C}(A)$ and $\alpha' \in \mathfrak{C}\left(A^{\perp}\right)$ then $\alpha \cap \alpha'$ has at most one element.

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- Observe that if $\alpha \in \mathfrak{C}(A)$ and $\alpha' \in \mathfrak{C}\left(A^{\perp}\right)$ then $\alpha \cap \alpha'$ has at most one element.
- Let A be a set: for all $\alpha, \alpha' \subseteq A$ write $\alpha \perp \alpha'$ iff $\#(\alpha \cap \alpha') \leq 1$.
- If $\mathfrak{A} \subseteq \mathfrak{P}(A)$, let $\mathfrak{A}^{\perp} = \{\alpha' \subseteq A;\ \alpha \perp \alpha',\ \text{for all } \alpha \in \mathfrak{A}\}$.

Orthogonality

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- Observe that if $\alpha \in \mathfrak{C}(A)$ and $\alpha' \in \mathfrak{C}\left(A^\perp\right)$ then $\alpha \cap \alpha'$ has at most one element.
- Let A be a set: for all $\alpha, \alpha' \subseteq A$ write $\alpha \perp \alpha'$ iff $\#(\alpha \cap \alpha') \leq 1$.
- If $\mathfrak{A} \subseteq \mathfrak{P}(A)$, let $\mathfrak{A}^\perp = \{\alpha' \subseteq A;\ \alpha \perp \alpha',\ \text{for all } \alpha \in \mathfrak{A}\}$.
- Observe that $\alpha' \in \mathfrak{A}^\perp$ iff for all $a'_1, a'_2 \in \alpha'$, $\{a'_1, a'_2\} \in \mathfrak{A}^\perp$.
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Characterization of morphisms

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- The coherence space model is thus derived from the relational model by orthogonality.
- This kind of construction generalizes to other notions of orthogonality, and to a generic categorical framework.

Perspectives

- Systematic categorical descriptions of models of LL exist
  - I was just lazy (see, *e.g.*, Melliès, Panoramas et Synthèses 2009)
  - one always recover a model of λ-calculus (co-Kleisli: $!A \multimap B$)
- Characterization of computational properties
  - normalizability
  - bounds on the number of cut elimination step
  - connexion with interaction types
  - rely on models of pure λ-calculus: solve $o = !o \multimap o$
- Quantitative semantics
  - can be described explicitly in a standard setting (Ehrhard's finiteness spaces)
  - differential linear logic
  - a guide for introducing and studying new models (see Michele Pagani's talk during the workshop)
- Duality given by interaction
  - at the core of geometry of interaction (operational-ish semantics)
  - the basis of game semantics, a multi-purpose family of models
- ...

Have a nice LL2016

Tutorial time!