Denotational semantics of linear logic

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Exercise 1 (Warmup) *Describe the relational semantics of*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,ax}{\vdash ?A^\perp, A}\,?d \qquad \dfrac{\dfrac{}{\vdash B^\perp, B}\,ax}{\vdash ?B^\perp, B}\,?d}{\vdash ?A^\perp, ?B^\perp, A \otimes B}\,\otimes}{\vdash ?A^\perp, ?B^\perp, !(A \otimes B)}\,!}{\vdash ?A^\perp \parr ?B^\perp, !(A \otimes B)}\,\parr$$

Does it define an isomorphism between $[\![!A \otimes !B]\!]$ *and* $[\![!(A \otimes B)]\!]$*?*

Solution: All the experiments of the subproof of the promotion rule are of the form:

$$\dfrac{\dfrac{\dfrac{}{\vdash \overset{a}{A^\perp}, \overset{a}{A}}\,\text{ax}}{\vdash \overset{[a]}{?A^\perp}, \overset{a}{A}}\,?d \qquad \dfrac{\dfrac{}{\vdash \overset{b}{B^\perp}, \overset{b}{B}}\,\text{ax}}{\vdash \overset{[b]}{?B^\perp}, \overset{b}{B}}\,?d}{\vdash \overset{[a]}{?A^\perp}, \overset{[b]}{?B^\perp}, \overset{(a,b)}{A \otimes B}}\,\otimes$$

An annotation for promotion is given by a finite list of those

$$\left(\dfrac{\dfrac{\dfrac{}{\vdash \overset{a_i}{A^\perp}, \overset{a_i}{A}}\,\text{ax}}{\vdash \overset{[a_i]}{?A^\perp}, \overset{a_i}{A}}\,?d \qquad \dfrac{\dfrac{}{\vdash \overset{b_i}{B^\perp}, \overset{b_i}{B}}\,\text{ax}}{\vdash \overset{[b_i]}{?B^\perp}, \overset{b_i}{B}}\,?d}{\vdash \overset{[a_i]}{?A^\perp}, \overset{[b_i]}{?B^\perp}, \overset{(a_i,b_i)}{A \otimes B}}\,\otimes\right)_{i=1,\dots,n}$$

and we can derive

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \overset{a_1}{A^\perp}, \overset{a_1}{A}}\,\text{ax}}{\vdash \overset{[a_1]}{?A^\perp}, \overset{a_1}{A}}\,?d \quad \dfrac{\dfrac{}{\vdash \overset{b_1}{B^\perp}, \overset{b_1}{B}}\,\text{ax}}{\vdash \overset{[b_1]}{?B^\perp}, \overset{b_1}{B}}\,?d}{\vdash \overset{[a_1]}{?A^\perp}, \overset{[b_1]}{?B^\perp}, \overset{(a_1,b_1)}{A \otimes B}}\,\otimes \quad \cdots \quad \dfrac{\dfrac{\dfrac{}{\vdash \overset{a_n}{A^\perp}, \overset{a_n}{A}}\,\text{ax}}{\vdash \overset{[a_n]}{?A^\perp}, \overset{a_n}{A}}\,?d \quad \dfrac{\dfrac{}{\vdash \overset{b_n}{B^\perp}, \overset{b_n}{B}}\,\text{ax}}{\vdash \overset{[b_n]}{?B^\perp}, \overset{b_n}{B}}\,?d}{\vdash \overset{[a_n]}{?A^\perp}, \overset{[b_n]}{?B^\perp}, \overset{(a_n,b_n)}{A \otimes B}}\,\otimes}{\vdash \overset{[a_1]+\cdots+[a_n]}{?A^\perp}, \overset{[b_1]+\cdots+[b_n]}{?B^\perp}, \overset{[(a_1,b_1),\dots,(a_n,b_n)]}{A \otimes B}}\,!}{\vdash \overset{([a_1,\dots,a_n],[b_1,\dots,b_n])}{?A^\perp \parr ?B^\perp}, \overset{[(a_1,b_1),\dots,(a_n,b_n)]}{A \otimes B}}\,\parr$$

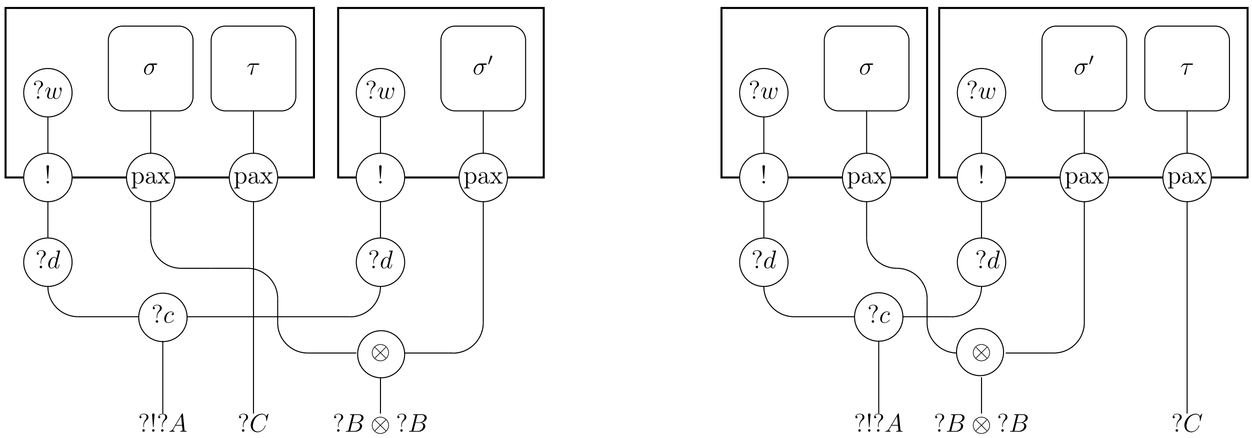
for each such family (here we write the n annotations as separate inference trees). Then we obtain the relation

$$\left\{\Big(([a_1,\cdots,a_n],[b_1,\cdots,b_n]),[(a_1,b_1),\ldots,(a_n,b_n)]\Big);\ n\in\mathbb{N},\ a_1,\ldots,a_n\in[\![A]\!],\ b_1,\ldots,b_n\in[\![B]\!]\right\}$$

which is clearly not the graph of a bijection between $[\![!A\otimes !B]\!]$ and $[\![!(A\otimes B)]\!]$ (the cardinalities of multisets on the left hand side are forced to match).

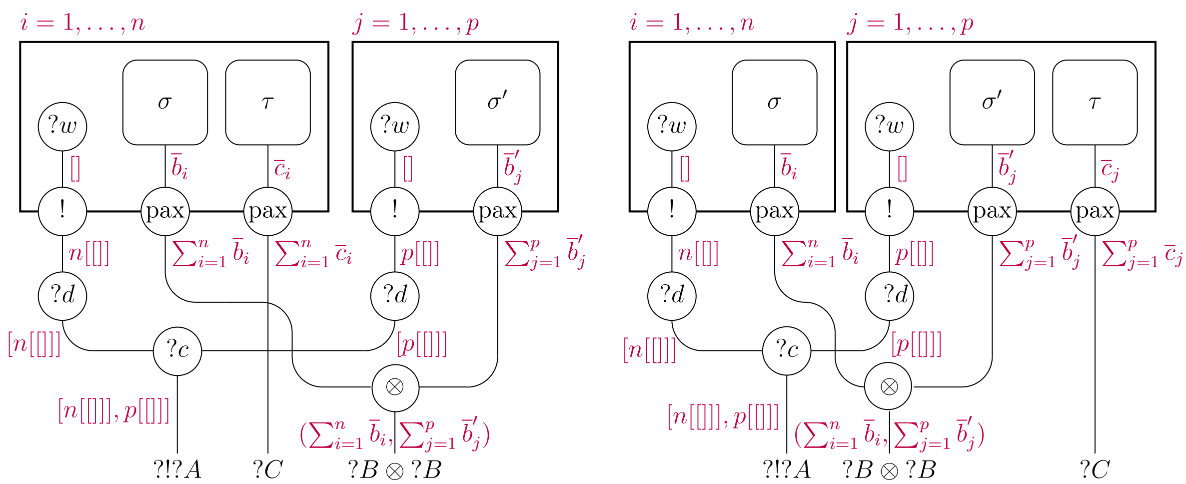
Fun fact: This is the monoidality of the comonad !.

Exercise 2 (Two MELL nets) *Compute the relational semantics and the coherence semantics of the following two MELL proof nets:*



where σ, σ' and τ are arbitrary proof nets with appropriate conclusions.

Solution: All possible experiments are of the following forms:



where $n[[]]$ denotes $[\underbrace{[],\ldots,[]}_{n}]$. So in the relational model, we obtain:

$$\left\{\left([n[[]],p[[]]],\left(\sum_{i=1}^{n}\bar{b}_i,\sum_{j=1}^{p}\bar{b}'_j\right),\sum_{i=1}^{n}\bar{c}_i\right);\right.$$
$$\left.n,p\in\mathbb{N},\ \bar{b}_1,\ldots,\bar{b}_n\in[\![\sigma]\!],\ \bar{b}'_1,\ldots,\bar{b}'_p\in[\![\sigma']\!],\ \bar{c}_1,\ldots,\bar{c}_n\in[\![\tau]\!]\right\}$$

on the left and

$$\Big\{ \Big(\big[n[[]], p[[]]\big], \Big(\textstyle\sum_{i=1}^n \bar{b}_i, \sum_{j=1}^p \bar{b}'_j \Big), \sum_{j=1}^p \bar{c}_j \Big) ; \quad n, p \in \mathbb{N},\ \bar{b}_1, \ldots, \bar{b}_n \in [\![\sigma]\!],\ \bar{b}'_1, \ldots, \bar{b}'_p \in [\![\sigma']\!],\ \bar{c}_1, \ldots, \bar{c}_p \in [\![\tau]\!] \Big\}$$

on the right.

In the coherence model, we must further require that each annotation of a formula $?D$ in the conclusion of a $(?c)$ or a (pax) node is an element of $|[\![D]\!]|$. So we must require, *e.g.*, that $\sum_{i=1}^n \bar{b}_i \in |[\![?B]\!]|$. More importantly, we need $[n[[]], p[[]]] \in |?!?A|$: this means $n[[]] \asymp_{!?A} p[[]]$. Obviously, $[] \frown_{?A} []$, hence $n[[]]$ and $p[[]] \in |!?A|$, and $n[[]] \frown_{!?A} p[[]]$: if we also have $n[[]] \asymp_{!?A} p[[]]$ then $n = p$.

We thus obtain the same semantics for both nets:

$$\Big\{ \Big(\big[n[[]], n[[]]\big], \Big(\textstyle\sum_{i=1}^n \bar{b}_i, \sum_{i=1}^n \bar{b}'_i \Big), \sum_{j=1}^n \bar{c}_i \Big) ; \quad n \in \mathbb{N},\ \bar{b}_1, \ldots, \bar{b}_n \in [\![\sigma]\!],\ \bar{b}'_1, \ldots, \bar{b}'_n \in [\![\sigma']\!],\ \bar{c}_1, \ldots, \bar{c}_n \in [\![\tau]\!], \quad \sum_{i=1}^n \bar{b}_i \in |?B|,\ \sum_{i=1}^n \bar{b}'_i \in |?B|,\ \sum_{i=1}^n \bar{c}_i \in |?C| \Big\}.$$

Exercise 3 (Rel **and** Coh**)** *In the following, we always chose interpretations of atoms such that $[\![X]\!]_{\mathrm{Rel}} = |[\![X]\!]_{\mathrm{Coh}}|$: it follows that $|[\![A]\!]_{\mathrm{Coh}}| \subseteq [\![A]\!]_{\mathrm{Rel}}$, for all formula A, and this inclusion is an identity whenever A is a formula of* MLL*. In this setting, whenever σ is a proof structure of conclusion Γ, both $[\![\sigma]\!]_{\mathrm{Rel}}$ and $[\![\sigma]\!]_{\mathrm{Coh}}$ are subsets of $[\![\Gamma]\!]_{\mathrm{Rel}}$.*

1. *Can you find a cut-free* MLL *proof structure whose relational semantics is always empty?*

   Solution: *No: experiments on cut-free* MLL *proof structures never fail. This was explained in the lecture about* MLL*.*

2. *Can you find a cut-free* MLL *proof structure whose relational semantics is not a clique?*

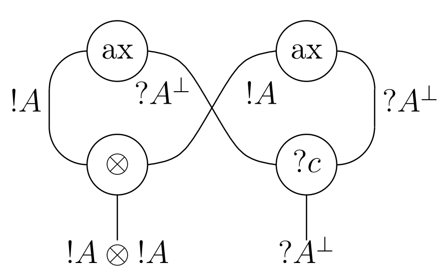
   Solution: *Yes: an example was given during the lecture on denotational semantics.*

3. *Is the coherence model injective on* MLL *proof nets?*

   Solution: *Yes: the semantics of proof nets is the same as that of any of the proof trees it represents. On* MLL *proofs trees, the interpretation in the coherence model is the same as that in the relational model; and the latter is injective.*

4. *Can you find an* MELL *proof net for which the coherence semantics and the relational semantics differ, without using promotion?*

   Solution: *Yes: just use a contraction node. For instance*



whose relational semantics contains all $((\bar{a}_1, \bar{a}_2), \bar{a}_1 + \bar{a}_2)$.

5. *Is the coherence model injective on* MELL *proof nets?*

   Solution: *No: this follows from the previous exercise.*

   Fun fact: *The injectivity of the relational model for* MELL *proof nets was a long standing conjecture formulated Lorenzo Tortora de Falco in the early 2000's. The above counter example to the injectivity of the coherence model is taken from one of Lorenzo's papers.*
   The conjecture has recently been solved (positively) by Daniel De Carvalho [arXiv:1502.02404].

Exercise 4 (Finiteness spaces) *For all* $\alpha, \beta \subseteq A$, *write* $\alpha \perp \beta$ *iff* $\alpha \cap \beta$ *is finite. Then if* $\mathfrak{A} \subseteq \mathfrak{P}(A)$, *write* $\mathfrak{A}^\perp = \{\alpha' \subseteq A;\ \alpha \perp \alpha' \text{ for all } \alpha \in \mathfrak{A}\}$.

1. *Show that* $\mathfrak{A} \subseteq \mathfrak{A}^{\perp\perp}$.

2. *Show that if* $\mathfrak{A} \subseteq \mathfrak{B}$ *then* $\mathfrak{B}^\perp \subseteq \mathfrak{A}^\perp$.

3. *Show that* $\mathfrak{A} = \mathfrak{A}^{\perp\perp}$ *iff* $\mathfrak{A}$ *is of the form* $\mathfrak{A}_0^\perp$ *for some* $\mathfrak{A}_0 \subseteq \mathfrak{P}(A)$.

4. *Have you really used the definition of the orthogonality relation in the above questions?*

   Solution: *The first three questions are pure abstract nonsense and we never really use the exact definition of* $\perp$.

A finiteness space A *is the data of a set* $|A|$ *and a set* $\mathfrak{F}(A)$ *of subsets of* $|A|$ *such that* $\mathfrak{F}(A) = \mathfrak{F}(A)^{\perp\perp}$.

5. *Show that* $\mathfrak{F}(A)$ *contains all finite subsets of* $|A|$.

   Solution: *If* α *is finite then* $\alpha \perp \alpha'$ *for all* α'.

6. *Show that if* $\alpha \subseteq \alpha' \in \mathfrak{F}(A)$ *then* $\alpha \in \mathfrak{F}(A)$.

   Solution: *If* $\alpha \subseteq \alpha'$ *and* $\alpha \perp \alpha''$ *then* $\alpha \perp \alpha''$.

7. *Show that if* α *and* $\alpha' \in \mathfrak{F}(A)$ *then* $\alpha \cup \alpha' \in \mathfrak{F}(A)$.

   Solution: *If* $\alpha \perp \alpha''$ *and* $\alpha' \perp \alpha''$ *then* $(\alpha \cup \alpha') \perp \alpha''$ *(unions of finite sets are finite).*

We say a relation $\varphi \subseteq |A| \times |B|$ *is finitary from* A *to* B *iff:*

- *if* $\alpha \in \mathfrak{F}(A)$ *then* $\varphi \cdot \alpha \in \mathfrak{F}(B)$;
- *if* $\beta' \in \mathfrak{F}(B)^\perp$ *then* $\varphi^\perp \cdot \beta' \in \mathfrak{F}(A)^\perp$;

where $\varphi \cdot \alpha$ *denotes relational application:*

$$b \in \varphi \cdot \alpha \text{ iff } (a, b) \in \varphi \text{ for some } a \in \alpha$$

and $\varphi^\perp \subseteq |B| \times |A|$ *is the reverse relation:*

$$(b, a) \in \varphi^\perp \text{ iff } (a, b) \in \varphi.$$

If A *and* B *are finiteness spaces then we define new finiteness spaces* $A^\perp$, $A \otimes B$ *and* $A \parr B$ *setting*

$$\begin{array}{rcl} \left|A^\perp\right| & = & |A| \\ \mathfrak{F}\left(A^\perp\right) & = & \mathfrak{F}(A)^\perp \\ |A \otimes B| & = & |A| \times |B| \\ \mathfrak{F}(A \otimes B) & = & \{\alpha \times \beta;\ \alpha \in \mathfrak{F}(A) \text{ and } \beta \in \mathfrak{F}(B)\}^{\perp\perp} \\ A \parr B & = & (A^\perp \otimes B^\perp)^\perp. \end{array}$$

8. *Prove that* $\varphi \subseteq |A| \times |B|$ *is finitary iff* $\varphi \in A^\perp \parr B$.

Solution: *Recall that* $A^\perp \parr B = ({}^\perp A \otimes B^\perp)$ *hence*

$$\mathfrak{F}\left(A^\perp \parr B\right) = \left\{\alpha \times \beta';\ \alpha \in \mathfrak{F}(A) \text{ and } \beta' \in \mathfrak{F}(B)^\perp\right\}^\perp .$$

We prove that φ *is finitary iff for all* $\alpha \in \mathfrak{F}(A)$ *and all* $\beta' \in \mathfrak{F}(B)^\perp$, $\varphi \perp \alpha \times \beta'$.

First assume φ *is finitary,* $\alpha \in \mathfrak{F}(A)$ *and* $\beta' \in \mathfrak{F}(B)^\perp$ *and write* $\varphi_0 = \varphi \cap \alpha \times \beta'$*: we prove* φ_0 *is finite. Take* $(a, b) \in \varphi_0$. *It is immediate that* $a \in \alpha$ *and* $b \in \beta'$. *It follows that* $b \in \varphi \cdot \alpha \in \mathfrak{F}(B)$ *and* $a \in \varphi^\perp \cdot \beta' \in \mathfrak{F}(A)^\perp$. *So* $\varphi_0 \subseteq (\alpha \cap (\varphi^\perp \cdot \beta') \times (\beta' \cap (\varphi \cdot \alpha))$, *which is finite.*

Now assume that, for all $\alpha \in \mathfrak{F}(A)$ *and all* $\beta' \in \mathfrak{F}(B)^\perp$, $\varphi \perp \alpha \times \beta'$. *For all* $\alpha \in \mathfrak{F}(A)$, *we prove* $\varphi \cdot \alpha \in \mathfrak{F}(B)$ *(the reverse direction is similar). For all* $\beta' \in \mathfrak{F}(B)^\perp$, *if* $b \in (\varphi \cdot \alpha) \cap \beta'$ *then there exists* $a \in \alpha$ *such that* $(a, b) \in \varphi$, *hence* $(a, b) \in \varphi \cap (\alpha \times \beta')$, *which is finite by assumption: it follows that* $\varphi \cdot \alpha \perp \beta'$.

9. *Fix a finiteness space* $[\![X]\!]_{\mathrm{Fin}}$ *for each propositional variable* X. *Then extend this semantics to* MLL *formulas in the obvious way and consider the relational model for* MLL *obtained by interpreting* X *by* $|[\![X]\!]_{\mathrm{Fin}}|$*: prove that for all* MLL *proof of* Γ, $[\![\pi]\!]_{\mathrm{Rel}} \in \mathfrak{F}([\![\Gamma]\!]_{\mathrm{Fin}})$.

Note: Focus on the tensor rule, the rest is bookkeeping.

Solution: *At this point you might as well have look at Thomas Ehrhard's paper (*Finiteness spaces. *Mathematical Structures in Computer Science, Cambridge University Press, 2005).*