Multiplicative Linear Logic

Olivier Laurent
LIP, CNRS - ENS Lyon, France

LL2016, Lyon
school: 7 and 8 November 2016

Exercise 1 (Starter)

1. *Compute* $((X^\perp \otimes Y) \parr (Y \parr X))^\perp$.
2. *Give a proof structure with conclusion* $(X \otimes (Y \otimes Y^\perp)) \parr X^\perp$.
3. *Is it a proof net?*
4. *Show that* $(A^\perp \parr B) \otimes (B^\perp \parr C) \vdash A^\perp \parr C$ *is provable.*

Exercise 2 (Beffara's formulas) *For each of the following three equivalences, give the two associated proof nets and two corresponding sequent calculus proofs:*

i. $A \dashv\vdash A \otimes (A^\perp \parr A)$

ii. $A \otimes (A^\perp \parr A) \dashv\vdash (A \otimes A^\perp) \parr A$

iii. $A \parr A^\perp \dashv\vdash (A \parr A^\perp) \otimes (A \parr A^\perp)$

Find A, B *and* C *such that* $(A \otimes B) \parr C \vdash A \otimes (B \parr C)$ *is* not *provable.*

Exercise 3 (Around Provability)

1. *Show that* $\vdash A \otimes B$ *is provable if and only if both* $\vdash A$ *and* $\vdash B$ *are provable.*
2. *Prove* $A \multimap B \dashv\vdash B^\perp \multimap A^\perp$
3. *Prove* $(A \otimes B) \multimap C \dashv\vdash A \multimap (B \multimap C)$
4. *Prove:* $A \dashv\vdash B$ *if and only if* $\vdash (A^\perp \parr B) \otimes (A \parr B^\perp)$ *is provable.*
5. *Describe how to transform any axiom rule* $\dfrac{}{\vdash A^\perp, A}\, ax$ *into a proof of* $\vdash A^\perp, A$ *using only axiom rules on atoms:* $\dfrac{}{\vdash X^\perp, X}\, ax$.

Exercise 4 (Proof Structures Manipulations)

1. *Given a proof structure with a unique conclusion* $A_1 \multimap B_1$ *and a proof structure with a unique conclusion* $A_2 \multimap B_2$, *explain how to build a proof structure with a unique conclusion* $(A_1 \otimes A_2) \multimap (B_1 \otimes B_2)$.
2. *Given a proof net with a unique conclusion* $A_1 \multimap B_1$ *and a proof net with a unique conclusion* $A_2 \multimap B_2$, *explain how to build a proof net with a unique conclusion* $(A_1 \otimes A_2) \multimap (B_1 \otimes B_2)$.

Exercise 5 (Deduction Property) *We call* deduction property *the following statement:*

$$A \vdash B \text{ is provable if and only if there exists a derivation } \begin{array}{c} \vdash A \\ \vdots \\ \vdash B \end{array}.$$

Discuss whether or not this property (or variants) holds in MLL.

Exercise 6 (Singleton Relational Model) *We consider the particular case of the relational model where each atom X is interpreted as a singleton $[\![X]\!] = \{\bullet\}$.*

1. *Compute the relational semantics of all formulas.*
2. *Compute the relational semantics of all sequents.*
3. *What are the possible values for the relational semantics of a proof?*

Exercise 7 (Semantics of Booleans) *We fix an atom X and we want to represent Booleans as proofs of $\vdash X^\perp, X \otimes X, X^\perp$.*

1. *Give all possible cut-free proof structures with conclusions $X^\perp, X \otimes X, X^\perp$.*
2. *We choose $[\![X]\!] = \{a, b\}$. Compute the relational semantics of each proof structure.*
3. *Which of the above proof structures are proof nets?*
4. *We call $\overline{\texttt{TRUE}}$ (resp. $\overline{\texttt{FALSE}}$) the proof net whose semantics contains $(a, (a, b), b)$ (resp. $(a, (b, a), b)$). Identify these two proof nets.*
5. *Build a proof net with conclusions $X^\perp \parr X^\perp, X \otimes X$ whose semantics contains $((a, b), (b, a))$. We call it ρ.*
6. *The proof structure σ is obtained from $\overline{\texttt{TRUE}}$ and ρ by introducing a cut between their dual conclusions. Prove σ is a proof net.*
7. *Compute the relational semantics of σ.*
8. *Eliminate cuts in σ and compute the relational semantics of the result.*
9. *Which function on Booleans does ρ represent?*
10. *Which other functions could be represented in the same manner using proof nets with conclusions $X^\perp \parr X^\perp, X \otimes X$?*